Stochastically Fluctuating Black-Hole Geometry, Hawking Radiation and the Trans-Planckian Problem

C. Barrabès[*,1], V. Frolov[†2,1] and R. Parentani[‡1]

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[1] *Laboratoire de Mathématiques et Physique Théorique, CNRS UPRES A 6083, Université de Tours, 37200 Tours, France*
[2] *Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1*

Abstract

We study the propagation of null rays and massless fields in a black hole fluctuating geometry. The metric fluctuations are induced by a small oscillating incoming flux of energy. The flux also induces black hole mass oscillations around its average value. We assume that the metric fluctuations are described by a statistical ensemble. The stochastic variables are the phases and the amplitudes of Fourier modes of the fluctuations. By averaging over these variables, we obtain an effective propagation for massless fields which is characterized by a critical length defined by the amplitude of the metric fluctuations: Smooth wave packets with respect to this length are not significantly affected when they are propagated forward in time. Concomitantly, we find that the asymptotic properties of Hawking radiation are not severely modified. However, backward propagated wave packets are dissipated by the metric fluctuations once their blue shifted frequency reaches the inverse critical length. All these properties bear many resemblences with those obtained in models for black hole radiation based on a modified dispersion relation. This strongly suggests that the physical origin of these models, which were introduced to confront the trans-Planckian problem, comes from the fluctuations of the black hole geometry.

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[*]e-mail: barrabes@celfi.phys.univ-tours.fr
[†]e-mail: frolov@phys.ualberta.ca
[‡]e-mail: parenta@celfi.phys.univ-tours.fr

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1 Introduction

In his original derivation of black hole radiance, Hawking [1] considered the propagation of a linear quantized field in a classical background geometry, that of a collapsing body. In this framework, one neglects two effects. One first neglects the gravitational back reaction effects, i.e. the consequences of the quantum response of the geometry to the energy density of the radiation field. To compute these effects is at present out of reach as it requires a better understanding of quantum gravity.

One also neglects the fluctuations of the geometry which are not due to the energy density of the radiation field. Besides quantum mechanical fluctuations of the gravitational field itself, there exist also metric fluctuations induced by the quantum fluctuations of other fields. The latter can be approximatively described by introducing stochastic noice sources in the right-hand side of the Einstein equations [2, 3, 4, 5]. In this description, one therefore deals with a stochastic ensemble of fluctuating geometries. Our aim is to study the propagation of a massless field in such an ensemble.

To describe metric fluctuations near the black hole horizon we shall use a model similar to that considered by York [6]. It is based on the hypothesis that the metric fluctuations are driven by a small oscillating flux of energy of an infalling null fluid. In our model, the metric fluctuations are represented by a linear superposition with different frequencies. Stochasticity will come into the picture by assuming that the amplitudes and phases of each mode are stochastic variables. Therefore to obtain the expectation value of any observable will require averaging over these variables. As we shall still neglect gravitational back reaction effects, we shall still have a linear field equation. In a former paper [7] , we analyzed the propagation of a test field in a metric characterized by a given classical fluctuation. Therefore, the novelty of the present work is to take an ensemble of such fluctuations.

The influence of black hole metric fluctuations on physical effects in the black hole geometry is an interesting and open problem. As we shall demonstrate in the paper, one of the justifications of considering a statistical ensemble of metric fluctuations is to confront the trans-Planckian problem[8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. This problem comes from the fact that low energy modes reaching $\mathcal{J}^+$ emerge from field configurations which possess, near the event horizon, arbitrarily high frequencies (as measured in a freely falling frame). Since gravitational interactions grow with the energy, one must call into question the validity of describing the propagation of these field configurations by free field theory. In particular, one might wonder if the low energy asymptotic properties of Hawking radiation, namely stationarity and thermality with a temperature determined by the surface gravity, are sensitive to the high energy behavior of the theory.

To answer this point, Unruh made an interesting proposal [8, 15]. He considered the propagation of sonic waves in an acoustic geometry governed by a wave equation whose dispersion relation is bended at high frequency. Indeed, only low frequency phonons propagate freely with a given velocity. For frequencies higher than a critical frequency ω_c, the dispersion relation is no longer linear and dissipation may occur. He then showed that the modification of the dispersion relation in no way affects the asymptotic properties of Hawking radiation as long as ω_c is much bigger than the surface gravity κ. The reason is that there is an adiabatic decoupling between these two energy scales. In [16], it was conjectured that light propagation near a black hole horizon should also be described by an effective mutilated theory and an alternative model "which may be more appropriate to

black hole physics" was proposed. The basic argument is that when a Hawking quantum is propagated backward close to the horizon, it will interact with a reservoir of modes, e.g. the high angular momentum modes which do not reach spatial infinity, or the metric fluctuations induced by these modes [18].

In this paper, we consider a stochastic ensemble[1] of metric fluctuations to describe these interactions. We shall show that light propagation in a stochastic metric indeed leads to an effective truncated theory near the event horizon. More precisely, we obtain the following. First the critical lenght ω_c^{-1} is determined by the amplitude of the metric fluctuations. Secondly, as far as forward propagation is concerned, the evolution of smooth wave packets (where smooth means that their in-frequency content is much below ω_c) is affected only slightly by the metric fluctuations. Thirdly, backward in time propagation of wave packets representing Hawking quanta is dramatically modified only when the blue shift factor brings their frequency close to ω_c. In this regime, the amplitude of the wave packet is rapidely dissipated (backward in time!).

At this point, the reader might wonder about the physical validity of these results since we do not know the precise nature of the metric fluctuations near the event horizon. In this respect, the following point should be emphasized. We are not studying the fluctuations themselves but only their effects on light propagation. As argued by Feynman and Hibbs[21], the effective propagation obtained by tracing out the degrees of freedom of the environment does not depend on the precise nature of the interactions of the test particle with it. For this universality to apply, one should neither ask too detailed questions (e.g. during too small time intervals) nor consider too strong interactions leading to significant recoils effects in the environment. The characterization of the domain of validity of the stochastic treatment is a complicated question which requires a detailed knowledge of the environment dynamics[22]. In the case of metric fluctuations, this is of course beyond our reach. But the crucial point is that, if there is an intermediate regime in which the particle weakly interacts with a large number of modes, it is *sufficient* to work with an appropriate simplified model. In our case, universality comes essentially from the exponentially growing Doppler effect encountered in backward propagation: how ever small is the critical length it will be reached in a logarithmicly short (advanced) time. The only condition it must obey is to be much smaller than the Schwarzschild radius.

The paper is organized as follow. In Section 2, we describe our fluctuating metric and we study how outgoing null rays propagate in it. The metric we choose results from the collapse of a massive spherically symmetric null shell which is followed by a small additional oscillating and infalling null flux.

In Section 3, we analyse the propagation of wave packets of a massless scalar field in an ensemble of fluctuating metrics. That is, we first obtain the propagation of a given initial wave packet in each realization of the geometry and then define the mean wave packet by averaging over the fluctuating part of the metric. Since the linearity of the problem is maintained, we formulate the problem as in a S-matrix language and show that this matrix is diagonal in Fourier components. From this equation one immediately sees that only the high frequency part of the spectrum is affected by the fluctuations.

In Section 4, we first analyse the Green function in the initial vacuum and show that the metric fluctuations do not significantly affect its short distance behavior thereby

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[1] There are several works in the literature wherein stochastic ensembles of black-hole metrics have been considered, see e.g. [19, 20]. However, to our knowledge, none of them confronts the role of ultra-high frequencies occuring in Hawking radiation.

guaranteeing unmodified properties of Hawking radiation. This is verified by computing directly the asymptotic flux of energy of quantum radiation.

In Section 5, we compare the backward propagations from $\mathcal{J}^+$ of a wave function describing a typical Hawking quantum in two cases: in our fluctuating metric and by modifying the dispersion relation as done in [15, 16]. The similarities are manisfest. We conclude the paper by making comments on the apparent breakdown of Lorentz invariance which appears in these effective theories.

2 Fluctuating Black-Hole Geometry

2.1 Metric ansatz

In this article we shall only consider spherical modes of metric fluctuations propagating in a spherically symmetric background. The most general spherical metric can be written in the form

$$dS^2 = \gamma_{AB}\,dx^A\,dx^B + R^2\,d\omega_2^2\,, \tag{2.1}$$

where $A, B = 0, 1$, and γ_{AB} and R are functions of x^A. The metric of a black hole of mass M formed by the collapse at $v = 0$ of a massive null shell with mass M is

$$dS^2 = (4M)^2 ds^2\,, \quad ds^2 = -A\,dv^2 + 2dv\,dr + r^2 d\omega_2^2\,, \tag{2.2}$$

where in the absence of fluctuations

$$A = A_0(r, v) = 1 - \frac{\vartheta(v)}{2r}\,, \tag{2.3}$$

$\vartheta(v)$ being the Heaviside step function equal to 1 for positive argument. For further convenience we have introduced the dimensionless coordinates (v, r), so that $R = 4Mr$ and $4Mv$ are the radius and the advanced time in units where $G = c = 1$. A conformal Penrose-Carter diagram of the spacetime is shown in Figure 1.

The most general metric perturbation preserving the form (2.1) of the metric is described by four functions of x^A: δr and $\delta\gamma_{AB}$. The remaining coordinate gauge freedom is generated by infinitesimal coordinate transformations $\xi^A(x)$. We fix the gauge by putting

$$\delta r = 0\,, \qquad \delta\gamma_{rr} = 0\,. \tag{2.4}$$

The perturbed metric can be written in the form

$$ds^2 = \Psi(-A\,dv^2 + 2dv\,dr) + r^2 d\omega_2^2\,, \tag{2.5}$$

with

$$\Psi = 1 + \delta\Psi\,, \qquad A = A_0 + \delta A\,. \tag{2.6}$$

Upon restricting attention to the propagation of radial null rays, the 2-dimensional conformal factor Ψ plays no role. In the following sections, we shall study the propagation of s-waves in the fluctuating black hole geometry. We shall demonstrate that, for s-modes, Ψ also drops out of the 4-dimensional Dalembertian. Hence only the function A will be relevant for us.
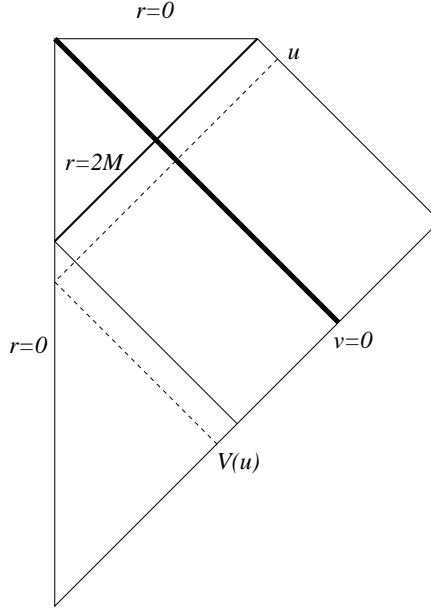
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Figure 1: Conformal Penrose-Carter diagram of the unperturbed collapsing black hole geometry. The massive (of mass M) null shell is represented by a heavy solid line at $v = 0$.

To further simplify the problem, we assume that the metric fluctuation δA is composed only of infalling radial null modes. Thus it is of the form

$$\delta A = -\frac{1}{2r}\,\vartheta(v)\,\mu(v)\,. \tag{2.7}$$

so that the perturbed metric is given by (2.5) with

$$A = A_0 + \delta A = 1 - \frac{\vartheta(v)[1 + \mu(v)]}{2r}\,. \tag{2.8}$$

For $\Psi = 1$ this is a Vaidya metric. The function $\mu(v)$ encodes the light-like infalling fluctuations. As in eq. (2.3), the step function in relation (2.8) indicates that the black hole results from the gravitational collapse at $v = 0$ of a massive null shell with mass M, and that there are no fluctuations prior to the collapse of the null shell. Therefore spacetime is flat to the past of the null shell.

2.2 Stochastic variables

To introduce the stochastic variables in simple terms, we postulate that $\mu(v)$ possesses a discrete[2] and non-degenerate Fourier decomposition:

$$\mu(v) = \sum_{\omega} \left[\mu_1^\omega \sin(\omega v) + \mu_2^\omega \cos(\omega v) \right] = \sum_{\omega} \mu_0^\omega \, \sin(\omega v + \phi_\omega)\,. \tag{2.9}$$

The equality is obtained by using polar coordinates in the $(\mu_1, \mu_2)-$plane

$$\mu_0^\omega = \sqrt{(\mu_1^\omega)^2 + (\mu_2^\omega)^2}\,, \quad \tan\phi_\omega = \frac{\mu_2^\omega}{\mu_1^\omega}\,. \tag{2.10}$$

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[2] A discrete spectrum arises for example in York's approach [6] based on the quasi-normal modes of the black-hole metric. However, the spectrum due to other fields can be continuous. The results of the present paper can be easily adopted to this case. It is sufficient to replace the discrete sum, $\sum_{\omega}$, by an integral, $\int d\omega\,\nu(\omega)$, where $\nu(\omega)$ is the number density of fluctuation modes.

We assume that the (real) amplitudes μ_1^ω and μ_2^ω are stochastic variables taking range from $-\infty$ to ∞. For simplicity we postulate that they are stochastically independent and that, for any frequency ω, the dispersions of μ_1^ω and μ_2^ω are equal. Thus, there is no preferred value of the phase ϕ_ω in the $(\mu_1^\omega, \mu_2^\omega)$ plane. In this case, $\tilde{\rho}_\omega(\mu_0^\omega)$, the distribution function for the amplitude μ_0^ω, satisfies the normalization condition

$$\int_0^{2\pi} d\phi_\omega \int_0^\infty d\mu_0^\omega \, \mu_0^\omega \, \tilde{\rho}_\omega(\mu_0^\omega) \, = \, 1 \, . \tag{2.11}$$

Later in the text, we shall assume that the amplitude μ_0^ω is a Gaussian variable whose distribution function is equal to

$$\tilde{\rho}_\omega(\mu_0^\omega) \, = \, \frac{1}{2\pi\tilde{\sigma}_\omega^2} \, \exp\left[-\frac{(\mu_0^\omega)^2}{2\tilde{\sigma}_\omega^2}\right] \, . \tag{2.12}$$

The coefficient $\tilde{\sigma}_\omega$ determines the dispersion of the amplitude μ_0^ω:

$$< (\mu_0^\omega)^2 >_\omega = \, 2\tilde{\sigma}_\omega^2 \, , \tag{2.13}$$

where $<>_\omega$ represents the average calculated with the distribution $\tilde{\rho}_\omega(\mu_0^\omega)$.

In what follows we shall have to deal with observables depending on the fluctuating geometry which obey the following factorization condition

$$Q = \prod_\omega Q_\omega(\mu_0^\omega, \phi_\omega) \, . \tag{2.14}$$

For these observables, we can consider each sector labeled by ω separately. It is then usefull to introduce the successive averages:

$$\bar{Q}_\omega(\mu_0^\omega) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_\omega \, Q_\omega(\mu_0^\omega, \phi_\omega) \, , \tag{2.15}$$

$$<Q_\omega>_\omega = 2\pi \int_0^\infty d\mu_0^\omega \, \mu_0^\omega \, \tilde{\rho}_\omega(\mu_0^\omega) \, \bar{Q}_\omega(\mu_0^\omega) \, , \tag{2.16}$$

$$\ll Q \gg = \prod_\omega <Q_\omega>_\omega \, . \tag{2.17}$$

The first equality gives $\bar{Q}_\omega$, the average of Q_ω over the stochastic phase ϕ_ω. The second one gives the result of averaging $\bar{Q}_\omega$ over the amplitude μ_0^ω. Finally, the overall ensemble average of the observable Q, is given by (2.17). The order in this averaging procedure follows from the fact that to perform the first average, one simply has to assume that there is no prefered direction in ϕ_ω. For the second instead, we need to know the distribution $\tilde{\rho}_\omega$. And for the third one, we must know the whole spectrum.

To determine the actual physical spectrum of metric fluctuations around a black hole horizon is an extremely complicated problem requiring a theory of quantum gravity, see [6, 18] for attemps to characterize this spectrum. In the present paper we shall not use any specific form of the spectrum. As we already mentioned, the effects of the metric fluctuations hardly depend on its exact form. That this is the case will appear progressively in the paper. Our only assumption is that the amplitude of metric fluctuations are much smaller than the gravitational radius of the black hole. In our dimensionless units, this gives $\tilde{\sigma} \ll 1$. This should be true for black holes of mass M much greater than the

Planckin mass m_{Pl} (in [6], the estimate dimensionless amplitude scales as $\tilde{\sigma} \sim (m_{\mathrm{Pl}}/M)$, whereas in [18] $\tilde{\sigma} \sim (m_{\mathrm{Pl}}/M)^{4/3}$).

Because of the factorizability of the operators and the statistical independence of the amplitudes, it will be sufficient to consider only a single fluctuation mode. To simplify the notations we shall drop the index ω in the amplitude and in the phase. That is we shall work in the fluctuating geometry (2.2), (2.8) with

$$\mu(v) = \mu_0 \sin(\omega v + \phi), \tag{2.18}$$

with $\mu_0 \ll 1$. We call the metric (2.2), (2.8) with $\mu(v)$ given by (2.18) a realization of the fluctuating geometry. By averaging over ϕ and μ_0 we thus assume that we are dealing with an ensemble of such realizations.

Finally, it should be stressed that ϕ, the phase of the fluctuations, has physical meaning. Indeed, the gravitational collapse of the null shell singles out a special moment of time (e.g. the moment of formation of the horizon), and fluctuations with different phases are not equivalent. Therefore ϕ is a stochastic variable and when computing the statistical average, integration over it is to be done.

2.3 Null ray propagation in a fluctuating geometry

To characterize the propagation of radial null rays in a given realization of the geometry we focus on the relation

$$v = V_\phi(u) \tag{2.19}$$

between the moment v of advanced time when the null ray was emitted from $\mathcal{J}^-$ and the retarded time u when it reaches $\mathcal{J}^+$. The above equation contains a simplified notation since $V_\phi(u)$ also depends on the amplitude μ_0 and the frequency ω – this is evident in eq. (2.20) below.

In the late time regime, i.e. $u \gg 1$, we have (for details, see [7])

$$V_\phi(u) = - \left[1 + \frac{\mu_0}{\sqrt{1+\omega^2}} \sin(\phi + \phi_0) + \right.$$

$$\left. e^{-u} \left(1 + \frac{\mu_0}{\omega} \cos(\phi + 2\phi_0) + \frac{\mu_0\, q(\omega)}{\sqrt{1+\omega^2}} \sin(\phi + \phi_0 + \omega u - \varphi_\Gamma(\omega)) \right) \right] + O(\mu_0^{\,2}), \tag{2.20}$$

where $\phi_0 = \arctan\omega$ and where $\varphi_\Gamma(\omega)$ and $q(\omega)$ only depend on the frequency of fluctuations. In this Section and the next one, we shall work with a simplified version of $V_\phi(u)$: Since the last two terms inside the large parenthesis in (2.20) are much smaller than 1 ($\mu_0 \ll 1$), we shall simply omit them. In Section 4 instead, we shall take them into account and also include all quadratic terms in μ_0.

The simplified version of relations (2.19) and (2.20) is of the form

$$w = W_\phi(u), \tag{2.21}$$

$$W_\phi(u) = w_0 \sin(\phi + \phi_0) + e^{-u}, \tag{2.22}$$

where we have introduced for later convenience

$$w = -1 - v, \quad w_0 = \frac{\mu_0}{\sqrt{1+\omega^2}}. \tag{2.23}$$

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For a given realization of the geometry and to first order in μ_0 the event horizon is given by the equation (see eq. (3.10) in [7])

$$r_\phi^{\text{EH}}(v) = \frac{1}{2}[1 + w_0 \sin(\omega v + \phi + \phi_0)] \,. \tag{2.24}$$

It crosses the collapsing null shell, $v = 0$ or $w = -1$, at the radius

$$r_\phi^{\text{EH}}(0) = \frac{1}{2}[1 + w_0 \sin(\phi + \phi_0)] \,. \tag{2.25}$$

Being traced backward in time the null geodesic giving rise to the horizon enters the flat spacetime region inside the collapsing shell, bounces at $r = 0$, and finally reaches $\mathcal{J}^-$ with w (defined by (2.23)) lying in the domain

$$w \in (-w_0, w_0) \,. \tag{2.26}$$

Therefore, for all ϕ, radial null rays emitted from $\mathcal{J}^-$ at advanced time $w < -w_0$ fall into the singularity $r = 0$. On the contrary, radial null rays emitted with $w > w_0$ always reach $\mathcal{J}^+$. The time of their arrival to $\mathcal{J}^+$ lies in the interval

$$u \in (u_-, u_+) \,, \tag{2.27}$$

where $u_\pm$ are

$$u_\pm = -\ln(w \mp w_0) \,. \tag{2.28}$$

Finally, the rays emitted in the interval $-w_0 < w < w_0$ reach $\mathcal{J}^+$ only for some values of ϕ. Since for w in this interval there always exists a phase such that the ray propagates along the horizon, the moment of arrival at $\mathcal{J}^+$ varies from $u_- = -\ln(w + w_0)$ to $u_+ = \infty$.

Figure 2 shows the behavior of radial null rays in the fluctuating black hole geometry characterized by a single mode of frequency ω. Ray **a** represents the event horizon, while rays **b** and **c** are rays which reach $\mathcal{J}^+$ or remain trapped inside the horizon, respectively. When they are traced back in time all these rays pass through the center of symmetry $r = 0$, and go to $\mathcal{J}^-$ along lines $v = \textit{const}$. It should be noted that the conformal Penrose-Carter diagrams for any given realization of the fluctuating black hole geometry is similar to the one shown in Figure 1 even though the event horizon no longer obeys the equation $r = 1/2$, see eq. (2.24).

3   Wave Propagation in a Fluctuating Geometry

3.1 δ-pulse propagation

3.1.1   Wave propagation in a given realization of the geometry

In this article, we study only s-modes of a minimally coupled massless scalar field χ. We introduce as usual $\varphi = r\chi$. Then, the four-dimensional Dalembertian equation $\Box\chi = 0$ when computed in our fluctuating metric (2.5) gives, see e.g [23],

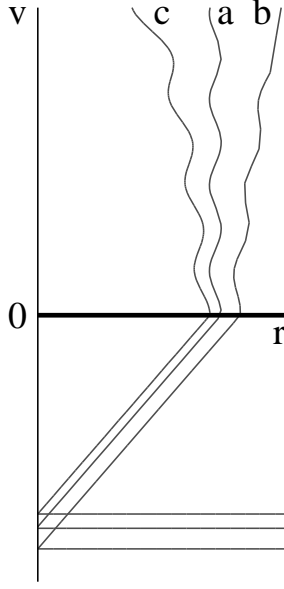$$\left[{}^{(2)}\Box - \frac{\partial_r A}{r} \right] \varphi = 0 \tag{3.1}$$

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Figure 2: Null rays propagation in a fluctuating geometry of the black hole.

Hence Ψ defined in eq. (2.5) plays no role. Moreover, upon neglecting the centrifugal quantum potential, one obtains the equation

$$^{(2)}\Box\varphi = 2\partial_v\partial_r\varphi + \partial_r(A\partial_r\varphi) = 0 \,. \tag{3.2}$$

When adopting this equation, one works in the geometrical optics approximation. Therefore, the spherically symmetric perturbations of the metric affect the global properties of the solutions of (3.2) only through the glueing of the null characteristics encoded in $V_\phi(u)$ given in eq. (2.22). By "global properties" we mean properties which can be measured on $\mathcal{J}^+$. We shall not compute the local value of the field near the event horizon in the fluctuating geometry. This would require the knowledge of the local description of outgoing null geodesics $u_\phi(v, r)$ for arbitrary v and r. Even though we restrict ourselves to global aspects, we shall see that the notion of a smeared horizon naturaly emerges. This should cause no surprise since the very definition of event horizon is global.

We shall denote the value of the solutions of (3.2) on $\mathcal{J}^\pm$ by a capital letter

$$\Phi^\pm = \varphi|_{\mathcal{J}^\pm} \,.$$

Using the coordinate w defined in (2.23) we call $\Phi^-(w)$ the initial value (or image) of the solution φ on $\mathcal{J}^-$, and $\Phi^+(u)$ the final value (or image) of φ on $\mathcal{J}^+$. For a fixed geometry, i.e. for a fixed μ_0 and ϕ, the knowledge of $\Phi^-(w)$ uniquely determines its image on $\mathcal{J}^+$. However, for backward propagation from $\mathcal{J}^+$ to $\mathcal{J}^-$, this is not true since $\mathcal{J}^+$ is not a complete Cauchy surface.

Let us consider what happens to a wave packet propagating in the black-hole fluctuating geometry. The wave packet is sent from $\mathcal{J}^-$ where it has image $\Phi^-(w)$. Because of the linearity of the problem, it is sufficient to know the evolution of the following δ-like pulse sent from $\mathcal{J}^-$

$$\Delta^-(w|w') = \delta(w - w') \,. \tag{3.3}$$

We call $\Delta_\phi^+(u|w')$ the image of this pulse on $\mathcal{J}^+$ in a given realization characterized by the value of ϕ. Since we work in the geometrical optics approximation, it is equal to

$$\Delta_\phi^+(u|w') = \delta(W_\phi(u) - w') \,, \tag{3.4}$$

where $W_\phi(u)$ is the value of w at which a radial null ray has to be sent from $\mathcal{J}^-$ in order to reach $\mathcal{J}^+$ at retarded time u.

In terms of Δ_ϕ^+, the image of the wave packet on $\mathcal{J}^+$ in this realization is given by

$$\Phi_\phi^+(u) = \int_{-\infty}^{\infty} dw \, \Phi^-(w) \, \Delta_\phi^+(u|w) \,. \tag{3.5}$$

3.1.2   Averaging over the phase

Using the notation introduced in eq. (2.15) we denote by $\bar{\Delta}^+(u|w)$ the average of $\Delta_\phi^+(u|w)$ over the phase. Throughout the paper the "bar" over a quantity indicates that this average has been taken. In terms of $\bar{\Delta}^+$, the average image of the wave packet on $\mathcal{J}^+$, is given by

$$\bar{\Phi}^+(u) = \int_{-\infty}^{\infty} dw \, \Phi^-(w) \, \bar{\Delta}^+(u|w) \,. \tag{3.6}$$

Thus our primary goal is to calculate $\bar{\Delta}^+(u|w)$. By definition it is given by

$$\bar{\Delta}^+(u|w) = \frac{1}{2\pi} \int_0^{2\pi} \delta(W_\phi(u) - w) \, d\phi \,. \tag{3.7}$$

We remind the reader that this equation follows from the hypothesis that all values of ϕ are equally probable, see section 2.2. To calculate this integral, we use the following property of δ-function:

$$\int_0^{2\pi} d\phi \, \delta(F(\phi)) = \sum_i \frac{1}{|F'(\phi_i)|} \,, \tag{3.8}$$

where ϕ_i are the roots of the equation $F(\phi) = 0$ lying in the interval $(0, 2\pi)$. That is, we simply need to find those solutions ϕ of the equation $W_\phi(u) = w$ which lie in the interval $(0, 2\pi)$.

Using the simplified expression (2.22) we obtain[3]

$$\bar{\Delta}^+(u|w) = \frac{\vartheta(w_0^2 - (e^{-u} - w)^2)}{\pi\sqrt{w_0^2 - (e^{-u} - w)^2}} \,. \tag{3.9}$$

For a fixed w, expression (3.9) determines $\bar{\Delta}^+(u|w)$ for $u \in (u_-, u_+)$, where

$$u_- = -\ln(w + w_0) \,, \tag{3.10}$$

$$u_+ = \begin{cases} -\ln(w - w_0) \,, & \text{for } w > w_0; \\ \infty \,, & \text{for } -w_0 < w < w_0. \end{cases} \tag{3.11}$$

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[3]Notice that $\bar{\Delta}^+$ is highly non linear in w_0. However it results from eq. (2.22) which has been linearized and simplified. Therefore, one must question the validity of its non-linear dependence in w_0. This question is addressed in Appendix A. The outcome of the analysis is that one can trust (3.9) in the non-linear regime since it correctly gives the dominant non-linear effects. The fact that the higher order terms in μ_0 of W_ϕ are all irrelevant indicates that there is an underlying universality of our results.

For $u \notin (u_-, u_+)$, $\bar{\Delta}^+(u|w)$ vanishes since no $\phi \in (0, 2\pi)$ satisfies $W_\phi(u) = w$. Similarly, for a fixed value of u, $\bar{\Delta}^+(u|w)$ does not vanish only for $w \in (-w_0 + e^{-u}, w_0 + e^{-u})$.

In a similar way we can consider propagation backward in time. It is again sufficient to make the calculations for a δ-like pulse. We call

$$^+\Delta(u|u') = \delta(u - u') \tag{3.12}$$

the image of the pulse on $\mathcal{J}^+$. Then, for a given realization of the geometry it has the following image on $\mathcal{J}^-$

$$^-\Delta_\phi(w|u') = \delta(u' - U_\phi(w)). \tag{3.13}$$

Using eq. (2.22), one has

$$U_\phi(w) = - \ln [w - w_0 \sin(\phi + \phi_0)]. \tag{3.14}$$

By averaging $^-\Delta_\phi$ over ϕ we get

$$^-\bar{\Delta}(w|u) = e^{-u} \bar{\Delta}^+(u|w), \tag{3.15}$$

where $\bar{\Delta}^+(u|w)$ is given in (3.9). The only difference is the Jacobian e^{-u} which relates the delta functions defined on $\mathcal{J}^-$ in terms of w and on $\mathcal{J}^+$ in terms of u. Then, the image on $\mathcal{J}^-$ which results from the backward propagation of a wavepacket which has the image Φ^+ on $\mathcal{J}^+$ is given by

$$^-\bar{\Phi}(w) = \int_{-\infty}^{\infty} du\, \Phi^+(u)\, ^-\bar{\Delta}(w|u). \tag{3.16}$$

It should be noted that in order to get this relation we assumed that, for any realization of the geometry, no signal was propagating backward from black hole interior. This point will be further discussed in section 3.3.

3.1.3   Averaging over the amplitude

The average value of an observable over an ensemble of amplitudes of metric fluctuation μ_0 characterized by the distribution $\tilde{\rho}_\omega(\mu_0)$ is given by eq. (2.16). In order to compute the amplitude average of $\bar{\Delta}^+$, we need to know $\tilde{\rho}_\omega$. In the case of a Gaussian distribution (2.12), the average can be easily performed. Since the impact of the metric fluctuation on the field is expressed by $w_0 = \mu_0/\sqrt{1 + \omega^2}$ instead of the origin amplitude μ_0, we introduce the new distribution function

$$\rho_\omega(w_0) = \frac{1}{2\pi\sigma_\omega^2} \exp(-\frac{w_0^2}{2\sigma_\omega^2}), \tag{3.17}$$

where

$$\sigma_\omega = \frac{\tilde{\sigma}_\omega}{\sqrt{1 + \omega^2}}. \tag{3.18}$$

It is easy to check that

$$2\pi \int_0^\infty dw_0\, w_0 \rho_\omega(w_0) = 1, \quad < w_0^2 >_\omega = 2\sigma_\omega^2. \tag{3.19}$$

where the average is now taken over w_0 with the distribution function $\rho_\omega(w_0)$.

Using using (3.9) and (3.17), the average of $\bar{\Delta}^+$ gives

$$<\Delta^+(u|w)>_\omega = \frac{1}{\sigma_\omega\sqrt{2\pi}} \exp\left[-\frac{(e^{-u}-w)^2}{2\sigma_\omega^2}\right]. \tag{3.20}$$

Thus the Gaussian character of ρ_ω, the distribution of metric fluctuations, is preserved and now characterizes the distribution of light rays. The same result applies to backward propagation since one has $<^-\Delta(w|u)>_\omega = e^{-u} <\Delta^+(u,w)>_\omega$.

At fixed u, $<\Delta^+>_\omega$ gives the image on $\mathcal{J}^-$ of the given u-ray sent from $\mathcal{J}^+$. This image gives the probability that this ray reaches the interval dw centered on w. For $u = \infty$, $<\Delta^+>_\omega$ therefore gives the image of the event horizon. Since we get a Gaussian distribution centered at $w = 0$, the position corresponding to the horizon in the unperturbed geometry, this indicates that the horizon is *smeared* by the stochastic fluctuations. And eq. (3.20) shows that the global properties of the field propagation are sensitive to this smearing of the event horizon.

3.2   Scattering operator in a fluctuating black-hole geometry

The relations (3.4), (3.9) and (3.20) solve the classical scattering problem in our fluctuating geometry. We now demonstrate how this problem can be solved in a more formal way. This more formal approach will be very useful for understanding the modifications of the propagation induced by the metric fluctuations.

In order to simplify the algebra it is most convenient to introduce a new coordinate on $\mathcal{J}^+$

$$y = e^{-u}. \tag{3.21}$$

The reason is the following. In terms of y, eq. (2.22) becomes

$$w = W_\phi(y) \equiv y + w_0 \sin(\phi + \phi_0). \tag{3.22}$$

Therefore, the effect of the metric fluctuations is simply to shift y with respect to w. The image Φ^+ of a wavepacket on $\mathcal{J}^+$ can be considered as a function of u or of y. In order to avoid confusion, we shall keep the notation $\Phi^+(u)$ for the function of u and shall use the notation $\mathbf{\Phi}^+(y)$ whenever the image is considered as a function of y.

In terms of y, $\bar{\Delta}^+(u|w)$ given by (3.9) takes the form

$$\bar{\mathbf{\Delta}}^+(y|w) = \frac{1}{\pi}\frac{\vartheta(w_0^2-(y-w)^2)}{\sqrt{w_0^2-(y-w)^2}}, \tag{3.23}$$

and relation (3.6) becomes

$$\bar{\mathbf{\Phi}}^+(y) = \frac{1}{\pi}\int_{-w_0+y}^{w_0+y}\frac{dw\,\Phi^-(w)}{\sqrt{w_0^2-(y-w)^2}}. \tag{3.24}$$

According to our definition (3.21) the coordinate y takes only positive values. But we can use relation (3.24) to define $\bar{\mathbf{\Phi}}^+(y)$ (at least formally) as a function on the complete y axis[4]. In this case, eq. (3.24) defines a linear operator acting on the space of functions defined on the real axis. We call this operator $\bar{\mathsf{D}}$. Thus (3.24) takes the form

$$\bar{\mathbf{\Phi}}^+ = \bar{\mathsf{D}}\,\Phi^-. \tag{3.25}$$

---

[4]The geometrical meaning of negative values of y will be presented in the next subsection.

A formal representation for this operator can be easily obtained by exploiting the fact that eq. (3.22) is a linear map. Indeed, for any realization of the geometry we have

$$\mathbf{\Phi}_\phi^+(y) = \Phi^-(y + w_0 \sin(\phi + \phi_0)) = e^{w_0 \sin(\phi + \phi_0)\partial}\,\Phi^-(y)\,. \tag{3.26}$$

Here ∂ is the operator of differentiation with respect of the argument of the function. This relation can be rewritten as

$$\mathbf{\Phi}_\phi^+ = \mathsf{D}_\phi\,\Phi^-\,, \tag{3.27}$$

where

$$\mathsf{D}_\phi = e^{w_0 \sin(\phi + \phi_0)\partial}\,. \tag{3.28}$$

Notice that this shift operator bears some similarities with that introduced in [24].

Let us first perform the average over the phase. Using the integral representation of the Bessel function of zero index

$$J_0(b) = \frac{1}{\pi}\int_0^\pi d\phi\,e^{ib\cos\phi}\,, \tag{3.29}$$

we get

$$\bar{\mathsf{D}} = \int_0^{2\pi}\frac{d\phi}{2\pi}\mathsf{D}_\phi = J_0(-iw_0\partial)\,. \tag{3.30}$$

Let us now show that (3.23) is a direct consequence of (3.30). For this purpose it is convenient to adopt the Dirac notations and to write the functions $\bar{\mathbf{\Phi}}^+$ and Φ^- as ket-vectors $|\bar{\mathbf{\Phi}}^+\rangle$ and $|\Phi^-\rangle$, respectively. Then, in the "coordinate" representation, we have

$$\bar{\mathbf{\Phi}}^+(x) = \langle x|\bar{\mathbf{\Phi}}^+\rangle\,, \quad \Phi^-(x) = \langle x|\Phi^-\rangle\,, \tag{3.31}$$

and equation (3.25) takes the form

$$\bar{\mathbf{\Phi}}^+(x) = \int_{-\infty}^\infty dx'\langle x|\bar{\mathsf{D}}|x'\rangle\,\Phi^-(x')\,. \tag{3.32}$$

Relation (3.30) implies that the operator $\bar{\mathsf{D}}$ is diagonal in the "p-representation". This directly follows from the linearity of eq. (3.22). Explicitly one has

$$\langle p|\bar{\mathsf{D}}|p'\rangle = \delta(p - p')\,J_0(w_0 p)\,. \tag{3.33}$$

Then, using

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(ipx\right)\,, \tag{3.34}$$

we get

$$\langle x|\bar{\mathsf{D}}|x'\rangle = \int_{-\infty}^\infty dp\int_{-\infty}^\infty dp'\langle x|p\rangle\langle p|\bar{\mathsf{D}}|p'\rangle\langle p'|x'\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty dp\,e^{-i(x - x')p}\,J_0(w_0 p)\,. \tag{3.35}$$

Calculating the integral (see e.g. [36]) we get

$$\langle x|\bar{\mathsf{D}}|x'\rangle = \frac{1}{\pi}\frac{\vartheta(w_0^2 - (x - x')^2)}{\sqrt{w_0^2 - (x - x')^2}}\,. \tag{3.36}$$

Thus $\langle y|\bar{\mathsf{D}}|w\rangle$ equals $\bar{\boldsymbol{\Delta}}^+(y|w)$ given by (3.23).

Up to now we have only been dealing with the stochasticity connected with the phase ϕ. Let us discuss what happens when we average the operator $\bar{\mathsf{D}}$ over the amplitude of the metric fluctuations. We again assume that the probability distribution is Gaussian (3.17). Using the relation (see [26] p.393, eq. 13.3.1)

$$\int_0^\infty dz z e^{-z^2} J_0(\alpha z) = \frac{1}{2} e^{-\alpha^2/4} \,, \tag{3.37}$$

we get that the average of $\bar{\mathsf{D}}$ over the fluctuation amplitude is

$$<\mathsf{D}>_\omega = \exp\left(-\frac{\sigma_\omega^2 \partial^2}{2}\right) \,. \tag{3.38}$$

This simple form results from the linearity of the map (3.22) and the Gaussian character of $<\Delta^+(u|w)>_\omega$ (3.20).

The generalization of these results to a spectrum of metric fluctuations with different frequencies is straightforward. Indeed, since $<\mathsf{D}>_\omega$ has a simple exponential form, it enjoys the factorization property (2.11). Thus, using (2.17), its total ensemble average

$$\ll\mathsf{D}\gg = \exp\left(-\frac{\sigma_{\text{eff}}^2 \partial^2}{2}\right) \,, \tag{3.39}$$

where

$$\sigma_{\text{eff}}^2 = \sum_\omega \sigma_\omega^2 \,. \tag{3.40}$$

Eq. (3.40) shows that the effect of the whole spectrum of metric fluctuations is to give rise to a single length which weights higher order derivative terms. This shows that the details of the fluctuations spectrum play no significant role for the scattering operator $\ll\mathsf{D}\gg$.

In resume, finding the image Φ^+ on $\mathcal{J}^+$ of a wavepacket determined by its data Φ^- on $\mathcal{J}^-$ is a "classical" S-matrix problem. In the absence of metric fluctuations, the linear operator S_0 relating Φ^- and Φ^+ is trivial (equal to 1) if we use the coordinates y and w. In the presence of a given realization of the metric the operator is given by eq. (3.28). When one computes the "mean" propagation by considering a stochastic ensemble of metric fluctuations, this simple relation is modified and takes the form (3.30), (3.38) or (3.39) according to the ensemble of metric fluctuations that one considers.

The extremely simple form of the operators D in the "p"-representation allows one to make a few general observations. In particular, we have

$$\ll\boldsymbol{\Phi}^+(p)\gg \equiv \langle p|\ll\boldsymbol{\Phi}^+\gg\rangle = e^{-\sigma_{\text{eff}}^2 p^2/2} \Phi^-(p) \,. \tag{3.41}$$

Thus only the high frequency components (i.e. of the order of σ_{eff}^{-1} and greater) of the initial wave packet Φ^- are strongly affected by the fluctuations of the geometry. Therefore the forward propagation of any smooth (i.e. of Fourier content much below σ_{eff}^{-1}) wave packet defined on $\mathcal{J}^-$ will not be significantly affected by the metric fluctuations. In other words, for classical black hole physics, the metric fluctuations are irrelevant if, as indicated in [6, 18], their "mean" amplitude σ_{eff} is of the order of the Planck length or smaller than it.

3.3   Backward in time scattering

Eventhough eq. (3.22) is perfectly symmetric in w and y, backward propagation is dramatically affected by the metric fluctuations.

To settle the discussion, we first clarify the geometrical meaning of negative y. To ease this analysis, we start with backward propagation in the absence of fluctuations. In this case, for positive y, eq. (3.22) gives $w = y$. However, since $\mathcal{J}^+$ is not a complete Cauchy surface, we need to consider the union of $\mathcal{J}^+$ and the whole event horizon $u = \infty$ in order to have a complete Cauchy surface. Thus we must introduce a coordinate along the horizon. The simplest choice consists in considering the negative values of y defined again by $y = w$. Indeed the negative y axis so defined covers the horizon from $r = 0, w = 0$ inside the collapsing shell till $w = -\infty$. Thus, the real axis $y \in (-\infty, \infty)$ forms a complete Cauchy surface and the functions $\boldsymbol{\Phi}^+(y)$ determine their image $\Phi^-(w)$ on $\mathcal{J}^-$ for all w.

This procedure also applies for any given realization of the fluctuating geometry. Indeed, the event horizon, when continued backward for negative v in the inside flat geometry (2.3), reaches $r = 0$ at advanced w time $W_\phi(y = 0) = w_0 \sin(\phi + \phi_0)$. Therefore the negative half line $y \in (-\infty, 0)$ defined by eq. (3.22) still covers the whole horizon and $y \in (-\infty, \infty)$ forms a complete final Cauchy surface. Since this is valid for any realization, it is meaningful to use the coordinate $y \in (-\infty, \infty)$ after having averaged over ϕ.

For $y > 0$, $\boldsymbol{\Phi}^+(y)$ gives the value on $\mathcal{J}^+$, while for $y < 0$ it gives the value on the horizon. For regular $\boldsymbol{\Phi}^+(y)$, in virtue of the symmetrical role played by y and w in eq. (3.22), the averaged image on $\mathcal{J}^-$ is determined by the same scattering operator D which governed forward propagation. In the case of the full ensemble average, one has

$$\ll \Phi^- \gg = \ll \mathsf{D} \gg \boldsymbol{\Phi}^+ . \tag{3.42}$$

In Fourier transform with respect to w and y this gives

$$\ll \Phi^-(p) \gg = \exp\left(-\sigma_{\text{eff}}^2 p^2 / 2\right) \boldsymbol{\Phi}^+(p) . \tag{3.43}$$

Of special interest for studying Hawking radiation are the final images such that no incoming field emerges from the horizon for any realization of the geometry. They are of the simple form

$$\boldsymbol{\Phi}_{out}^+(y) = \vartheta(y) \boldsymbol{\Phi}^+(y) . \tag{3.44}$$

Unless $\boldsymbol{\Phi}^+(y)$ vanish sufficiently rapidly when $y \to 0$, these modes are singular at $y = 0$.

This problem will be studied in detail in Section 5. It reveals the important role played by the fluctuating horizon geometry for backward propagation. The asymmetry between backward and forward propagation comes from the fact that the inertial time on $\mathcal{J}^+$ which characterizes out-frequencies $\lambda = \partial_u$ is u and not $y = e^{-u}$. Then, the so defined out-frequencies are exponentially blue-shifted when propagated backward near the event horizon. This purely kinematical effect is at the origin of the trans-Planckian problem and has here dramatic consequences since higher derivative terms are present. Indeed, eq. (3.43) when applied to out-functions (3.44) which vanish for negative y gives, in the position representation,

$$\ll \Phi^-(w) \gg = \left[\exp\left(-\frac{1}{2}(\sigma_{\text{eff}}\, e^u \partial_u)^2\right)\, \Phi^+(u)\right]_{u=-\ln w} . \tag{3.45}$$

The dramatic consequences can now be seen: how ever smooth is the final data $\Phi^+(u)$, the fluctuations of the geometry will inevitably affect its backward propagation if it is centered

around a sufficiently late retarded time. Moreover if it does not vanish sufficiently fast (i.e. faster than[5] e^{-u}) when one approaches the horizon, the asymptotic behaviour of $\ll D \gg$ intervenes.

We shall further analyze these points in quantum mechanical terms in Section 5. Before presenting this material we shall show that the asymptotic properties of Hawking radiation are not significantly affected by the fluctuations when their average amplitude σ_{eff} is much smaller than the Schwarzschild radius ($= 1/2$ in our units).

4 Hawking Radiation

4.1 Green function in the in-vacuum

The simplest way to understand why the metric fluctuations do not significantly modify the asymptotic properties of Hawking radiation consists in analysing the Green function (more specifically, the positive frequency Wightman function[27]) evaluated in the initial vacuum state. Indeed, as shown in [28], when the short distance expansion of this function evaluated near the event horizon reduces to the standard (Hadamard) behavior, Hawking radiation obtains on $\mathcal{J}^+$.

In the absence of fluctuation, the in-vacuum is the vacuum state with respect to positive frequencies defined on $\mathcal{J}^-$ in terms of the inertial advanced time derivative $i\partial_v$. Therefore, for late time, the u-part of the Green function evaluated in this vacuum is controled by $V(u) = -1 - e^{-u}$. Explicitely, one has

$$
\begin{aligned}
G^{in}(u, u') &= \int_0^\infty \frac{dp}{4\pi p} e^{ip(V(u) - V(u') + i\epsilon)} \\
&= \frac{1}{4\pi} \ln(V(u) - V(u') + i\epsilon) = \frac{1}{4\pi} \ln(-e^{-u} + e^{-u'} + i\epsilon).
\end{aligned}
\tag{4.1}
$$

The thermal and steady character of Hawking radiation follows from the exponential relation between V and u, which remains unchanged for arbitrary large time u, and from the fact that for small V intervals ($\delta V \ll 1$), the in-Green function behaves like $\ln \delta V$. Therefore these are the two conditions which should be met in order to see if the metric fluctuations lead to modifications of Hawking radiation.

In a given realisation of the metric fluctuation, one simply replaces the unperturbed relation $V(u) = -1 - e^{-u}$ by its modified version eq. (2.20). Upon considering the ensemble of geometries, one averages the Green function over the ensemble. After averaging over phases, the mean function is thus given by

$$
\begin{aligned}
\bar{G}^{in}(u, u') &= \int_0^{2\pi} \frac{d\phi}{2\pi} G_\phi^{in}(u, u') \\
&= \frac{1}{8\pi^2} \int_0^{2\pi} d\phi \ln(V_\phi(u) - V_\phi(u') + i\epsilon).
\end{aligned}
\tag{4.2}
$$

---

[5]This condition on the decrease of the wave packet already appeared in the literature [29] in the Unruh detector context (see also [13] for its transposition in black hole physics). In the Unruh case, if one imposes that the fluxes of energy emitted by the accelerated detector be regular in an inertial frame, the coupling between this detector and the field must decrease faster than $e^{-a\tau}$ where a is the acceleration and τ the detector's proper time.

First, we notice that when using the simplified expression (2.22) Hawking radiation is not at all affected since the constant shift cancels out. The reason is clear: in each realization, two nearby points are shifted by almost the same amount, see Figure 2.

Therefore only higher order effects might affect Hawking radiation. To compute them, it is again usefull to use $y = \exp(-u)$. In terms of y, using eq. (2.20), one has

$$V_\phi(u) - V_\phi(u') = F_\phi(y') - F_\phi(y) + O(w_0^2) , \tag{4.3}$$

where

$$F_\phi(y) = y(1 + a\cos(\phi + 2\phi_0) + b\sin(\phi + \psi_0 - \omega \ln y)) . \tag{4.4}$$

Here $\psi_0 = \phi_0 - \varphi_\Gamma$. The explicit form of the coefficients a and b can be easily obtained from eq. (2.20). They are not important for us now. It is sufficient to mention that both these coefficients are of first order in w_0. For close points $y' = x + z/2$ and $y = x - z/2$ ($z \ll 1$) one has

$$F_\phi(y') - F_\phi(y) = z \left[1 + \sum_{n=0}^{\infty} c_n(\phi) z^{2n} \right] . \tag{4.5}$$

All the coefficients c_n are of the first order in w_0 since F_ϕ has been linearized. In particular, the first coefficient

$$c_0(\phi) = a\cos(\phi + 2\phi_0) + b\sin(\phi + \psi_0 - \omega \ln x) - b\omega\cos(\phi + \psi_0 - \omega \ln x) . \tag{4.6}$$

is a linear superposition of sine and cosine whose arguments are linear in ϕ. It is easy to check that this is also true for all other coefficients $c_n(\phi)$. Then, using (4.3) and (4.5), we can expand $\ln(V_\phi(u) - V_\phi(u') + i\epsilon)$ in powers of w_0 and conclude that after averaging over ϕ one has

$$\bar{G}^{in}(u, u') = G^{in}(u, u') + w_0^2 H(u, u') , \tag{4.7}$$

since the averaged value of all first order terms in w_0 vanishes. The crucial point is that $H(u, u')$ is finite when $u \to u'$. This implies that the corrections to Hawking radiation are at least of second order in w_0 and non-diverging in the late time regime. Upon considering the average over the fluctuation amplitudes the same result holds. These points are explicitly verified in the next section.

4.2 Quadratic corrections to Hawking Flux

The (quantum mechanical) mean energy flux of Hawking radiation measured at $\mathcal{J}^+$, dE/du, was obtained in [7] in the case of the oscillating Vaidya (2.8) metric with the phase ϕ in eq. (2.18) put equal to zero. The corrections were computed up to second order in the fluctuation amplitude μ_0 by using the 2D model based on eq. (3.2). In that case, the energy flux can be decomposed into the sum of a permanent part $(dE/du)^{\text{perm}}$ and a fluctuating part $(dE/du)^{\text{fluct}}$

$$dE/du = (dE/du)^{\text{perm}} + (dE/du)^{\text{fluct}} . \tag{4.8}$$

The permanent part is equal to

$$(dE/du)^{\text{perm}} = \frac{\kappa^2}{48\pi} \left[1 + \frac{1}{2} w_0^2(1 + \omega^2)q^2(\omega) + w_0^2 \right] , \tag{4.9}$$

where $\kappa = (8\pi M)^{-1}$ is the surface gravity of the unperturbed black hole, and

$$q(\omega) = \frac{\sqrt{2\pi}}{\sqrt{\omega(e^{2\pi\omega} - 1)}}. \tag{4.10}$$

In the absence of metric fluctuations, the permanent part of the flux reduces to its usual value, $\kappa^2/48\pi$. The corrections due to these fluctuations are second order in w_0. The fluctuating part $(dE/du)^{\text{fluct}}$ is a linear combination of terms oscillating with frequency ω and 2ω and, by definition, vanishes when integrated over the retarded time u.

We shall now compute this flux in a stochastic metric by averaging its value over the phase and amplitude of the metric fluctuations. We first study the influence of a single stochastic phase ϕ. In particular we shall check that averaging over this phase *and* over u does not affect the value of permanent part of the flux. For this purpose, as in [7], we calculate dE/du from the 2D expression

$$\frac{dE}{du}(u, \phi) = \frac{\kappa^2}{12\pi}(dV_\phi/du)^{1/2}\frac{d^2}{du^2}\left[(dV_\phi/du)^{-1/2}\right], \tag{4.11}$$

where $V_\phi(u)$ now depends on the phase ϕ, see (2.20). Since we want to compute the quadratic corrections in w_0, we need to know $V_\phi(u)$ up to second order in w_0. In the late time regime and to this order in w_0, $V_\phi(u)$ can be decomposed according to the dependence of the various terms in u. Using eqs. (4.51-60) in [7], one gets

$$V_\phi(u) = V_0(\phi) - e^{-u}\left[V_1(\phi) + V_2(u, \phi) - \frac{w_0^2}{2}u\right]. \tag{4.12}$$

By definition V_0 and V_1 do not depend on u. Note that V_0, V_1 and V_2 parametrically depend on the frequency ω of the fluctuations.

For a given ϕ, i.e. for a given realization of the geometry, one easily sees from (4.11) that dE/du does not depend on the values of V_0 and V_1. Therefore one is free to put $V_0 = 0$ and $V_1 = 1$ and to work with $V_\phi(u)$ as

$$V_\phi(u) = -e^{-u}\left[1 + V_2(u, \phi) - \frac{w_0^2}{2}u\right]. \tag{4.13}$$

The novelty with respect to eq. (2.20) arises from the fact that $V_\phi(u)$ is now developed up to second order in w_0. As $V_2(u, \phi)$ has no 0-th order term in w_0 it can be decomposed as

$$V_2(u, \phi) = w_0 V_{2,1}(u, \phi) + w_0^2 V_{2,2}(u, \phi). \tag{4.14}$$

The first term is already known, see eq. (2.20),

$$V_{2,1}(u, \phi) = q(\omega)\sin(\phi + \phi_0 + \omega u - \varphi_\Gamma), \tag{4.15}$$

where $\varphi_\Gamma(\omega)$ is a phase independent of u. $V_{2,2}(u, \phi)$ is a complicated expression containing oscillating terms with respect to u and ϕ. It is of little interest to write it explicitly. In what follows we shall only need its averaged value with respect to ϕ since we are interested by the expression of the averaged energy flux to second order in w_0.

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In terms of these two functions, the energy flux reads

$$
\frac{dE}{du}(u,\phi) = \frac{\kappa^2}{48\pi}\Big[1 + w_0^2 - 2w_0(V'_{2,1} - V''_{2,1}) + 2w_0^2(V_{2,1} - V'_{2,1})(V'_{2,1} - V'''_{2,1})
$$
$$
+ \quad 3w_0^2(V'_{2,1} - V''_{2,1})^2 - 2w_0^2(V'_{2,2} - V'''_{2,2})\Big]\,, \tag{4.16}
$$

where $'$ stands for $\partial/\partial u$.

Using as in Section 2 a "bar" to denote averaging over ϕ one gets

$$
\overline{\frac{dE}{du}}(u) = \frac{\kappa^2}{48\pi}\Big[1 + w_0^2 + \frac{1}{2}\,w_0^2(1+\omega^2)q^2(\omega) - 2w_0^2\overline{(V'_{2,2} - V'''_{2,2})}\Big]\,. \tag{4.17}
$$

The first w_0^2 term was already present in eq. (4.9) and is due to the last term in (4.12). The second one comes from quadratic terms in $V_{2,1}$ given by (4.15). The last term depends on u and is equal to

$$
\overline{V'_{2,2} - V'''_{2,2}} = (1 - 2\omega^2)q(\omega)\cos(\omega u - \varphi_\Gamma) + \omega(3 - 4\omega)\sin(\omega u - \varphi_\Gamma)\,. \tag{4.18}
$$

These results show that after taking the averages over ϕ and over u one gets back the same permanent contribution to the Hawking mean flux as in a given realisation of the geometry, see eq. (4.8). The reason is that averaging over u erases all dependence in ϕ.

Eq. (4.17) can be easily extended to a spectrum of metric fluctuations according to the lines presented in section 2.2. Limiting ourselves to the permanent part of (4.17) we obtain

$$
\left(\overline{\frac{dE}{du}}\right)^{\text{perm}} = \frac{\kappa^2}{48\pi}\Big[1 + w_0^2\Big(1 + \frac{1}{2}\,(1+\omega^2)q^2(\omega)\Big)\Big]\,, \tag{4.19}
$$

which gives the average over the phase of a single fluctuating mode of frequency ω. When using the Gaussian distribution ρ_ω, see (3.17), the average over the amplitude $w_0(\omega)$ of the fluctuating mode gives

$$
<\left(\frac{dE}{du}\right)^{\text{perm}}>_\omega = \frac{\kappa^2}{48\pi}\Big[1 + 2\sigma_\omega^2\Big(1 + \frac{1}{2}\,(1+\omega^2)q^2(\omega)\Big)\Big]\,, \tag{4.20}
$$

where (3.19) has been used.

Upon considering the whole spectrum of fluctuations, we get

$$
\ll\left(\frac{dE}{du}\right)^{\text{perm}}\gg = \frac{\kappa^2}{48\pi}\Big[1 + 2\sum_\omega \sigma_\omega^2\Big(1 + \frac{1}{2}\,(1+\omega^2)q^2(\omega)\Big)\Big]\,. \tag{4.21}
$$

To perform this last summation requires the knowledge of the spectrum, here represented by the set of σ_ω.

5 Fluctuating Geometry and Trans-Planckian Problem

5.1 Hawking radiation for modified dispersion relations

In this Section we shall establish the close analogies between the backward propagation in a fluctuating metric and the altered propagations which have been recently studied

and which result from the modifications of the dispersion relation in the high frequency regime.

Before presenting the technical details, it is appropriate to recall the following points. These models have been introduced in order to show that the mutilation of the dispersion relation for frequencies higher than ω_c, which is the equivalent of σ_{eff}^{-1} in our case, in no way affect the (low energy) properties of Hawking radiation, namely stationarity and thermality. Following the original work of Unruh[8, 15], many models have been analysed. Their common property is that the Dalembertian is modified by the addition of higher derivative terms weighted by negative powers of ω_c. In these works, the modifications have been inspired by hydrodynamics[31], electrodynamics in a dielectric medium[32], field theory on a lattice theory[33], string theory[34] or by guessing what the physics near a horizon might be[16]. In all these models, the following properties obtain

1. Fowardly propagated wave packets are unaffected by the modification of the dispersion relation as long as their in-frequency content is much below the critical frequency ω_c.

2. No significant modifications of the asymptotic properties of Hawking radiation as long as the surface gravity satisfies $\kappa \ll \omega_c$.

3. Dramatic modifications of backward propagated late-time wave packets of out-frequency λ when the blue shifted value λe^u reaches ω_c.

It should be already clear to the reader that our effective propagation in a fluctuating metric possesses many similarities with these models. Our aim is now to establish the parallelism in simple and analytical terms. To this end we shall exploit the following facts.

First we exploit the stationarity of the unperturbed background geometry by considering the backward propagation of a plane wave of out-frequency λ defined on $\mathcal{J}^+$. We shall consider its image on $v = 0$ rather than on $\mathcal{J}^-$ in order to ignore the backward propagation from $v = 0$ till $\mathcal{J}^-$ which is very much dependent on a model of a collapsing body and presumably irrelevant for black hole physics. The relationship with what we did in the former Section is straightforward since w, defined on $\mathcal{J}^-$, corresponds to $2r - 1$ on $v = 0$ in the dimensionless coordinates defined in Section 1 where $\kappa = 1$.

Secondly, we work in Fourier transform with respect to w as it provides an elegant characterization of in-vacuum. Since r is an afine parameter along $v = \text{const}$, $-i\partial_r$ corresponds to positive frequencies measured by inertial observers when they cross the event horizon. Therefore, when infalling observers see no particles, it means that the state of the radiation field corresponds to vacuum with respect to frequency modes of positive $-i\partial_r$. Moreover, since $\partial_r|_{v=0} = 2\partial_w|_{\mathcal{J}^-}$, the initial vacuum with respect to w coincides with vacuum as seen by infalling observers, see [30, 10].

Lets now review how these concepts translate in mathematical terms and how they can be used to compute the consequences of modifying the dispersion relation at high frequencies. In this we shall present the "alternative" model of [16] for its simplicity and its generality.

The 2D Dalembertian (3.2) for out-going modes with out-energy λ propagating near the event horizon ($r - 1/2 \ll 1/2$) gives, in the unperturbed Vaydia metric (2.2),

$$(1 - 2r)i\partial_r\varphi_\lambda = 2i\partial_v\varphi_\lambda = 2\lambda\varphi_\lambda \,. \tag{5.22}$$

Along $v = 0$, it is useful to express r in terms of $w = 2r - 1$. This leads to the simplified equation

$$w\partial_w \varphi_\lambda = i\lambda \varphi_\lambda \,. \tag{5.23}$$

The general solution is of the form $\varphi_\lambda = A\vartheta(w)w^{i\lambda} + B\vartheta(-w)(-w)^{i\lambda}$. The normalized out-mode describing the one particle state of energy λ is given by

$$\varphi_\lambda^{out} = \vartheta(w)\frac{w^{i\lambda}}{\sqrt{4\pi\lambda}} \,. \tag{5.24}$$

On the other hand, the mode leaving on the other side of the horizon describes the partner of this Hawking quantum.

In Fourier transform $p = i\partial_w$ eq. (5.23) becomes

$$\partial_p\left(p\varphi_\lambda\right) = -i\lambda\varphi_\lambda \,. \tag{5.25}$$

In terms of p, the normalized in-mode φ_λ^{in} contains only positive p and is thus given by $\varphi_\lambda^{in} = \vartheta(p)p^{-i\lambda-1}/\sqrt{4\pi\lambda}$. To obtain the Bogoliubov coefficients encoding the thermal flux of outgoing quanta it suffices to inverse Fourier transform this mode and use the definition (5.24). Explicitely one has

$$
\begin{aligned}
\varphi_\lambda^{in}(w) &= \int_0^\infty \frac{dp}{\sqrt{2\pi}}e^{ipw}\varphi_\lambda^{in}(p) \\
&= \frac{\Gamma(-i\lambda)}{\sqrt{2\pi}}\frac{(\epsilon - iw)^{i\lambda}}{\sqrt{4\pi\lambda}} \\
&= \frac{\Gamma(-i\lambda)}{\sqrt{2\pi}}\left[e^{\pi\lambda/2}\varphi_\lambda^{out} + e^{-\pi\lambda/2}\vartheta(-w)\frac{(-w)^{i\lambda}}{\sqrt{4\pi\lambda}}\right] \,,
\end{aligned}
\tag{5.26}
$$

where $\Gamma(z)$ is the Euler gamma function and where ϵ is small and positive. It fixes the relative weights of $w^{i\lambda}$ for positive and negative real values[30], as shown explicitly in the third line. This relative weight determines in turn the ratio of the Bogoliubov coefficients α_λ and β_λ: $|\alpha_\lambda/\beta_\lambda| = e^{\pi\lambda}$.

For later purpose, we notice that the same result can also be obtained the other way around, by Fourier transforming the out-mode (5.24). In this case one gets

$$
\begin{aligned}
\varphi_\lambda^{out}(p) &= \int_0^\infty \frac{dw}{\sqrt{2\pi}}e^{-ipw}\varphi_\lambda^{out}(w) \\
&= \frac{\Gamma(i\lambda + 1)}{\sqrt{2\pi}}\frac{(\epsilon + ip)^{-i\lambda-1}}{\sqrt{4\pi\lambda}} \\
&= \frac{(-i)\Gamma(i\lambda + 1)}{\sqrt{2\pi}}\left[e^{\pi\lambda/2}\varphi_\lambda^{in} - e^{-\pi\lambda/2}\vartheta(-p)\frac{(-p)^{-i\lambda-1}}{\sqrt{4\pi\lambda}}\right] \,.
\end{aligned}
\tag{5.27}
$$

The only subtility concerns again the prescription given by ϵ. It arises this time from the fact that the final wave packet vanishes on the other side of the horizon. Upon Fourier transform, this fixes the relative weights on the positive and negative real p axis. This can be used again to determine the ratio of the Bogoliubov coefficients.

We are now in position to modify the dispersion relation. In a flat metric, this relation is simply $p = \lambda$. To modify it, we write it as $g(p) = \lambda$. The only condition that g must satisfy is that for small p is behaves as $g(p) = p(1 + O(p/\omega_c))$. Near the horizon, for

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$\omega_c \gg 1$ and up to a normal ordering ambiguity of ∂_p and $g(p)$ which plays no role in a WKB approximation, eq. (5.25) becomes

$$\partial_p \left(g(p)\tilde{\varphi}_\lambda \right) = -i\lambda\tilde{\varphi}_\lambda . \tag{5.28}$$

The general solution of this modified equation is given by

$$\tilde{\varphi}_\lambda = A\vartheta(p)\frac{e^{-i\lambda \int^p dp'/g(p')}}{g(p)} + B\vartheta(-p)\frac{e^{-i\lambda \int^p dp'/g(p')}}{g(p)} . \tag{5.29}$$

In these terms, the three properties listed above are easily obtained. The first one follows from the definition of $g(p)$ which deviates from linearity only for $p > \omega_c$. The second point is verified by using the fact that in-modes characterizing vacuum for infalling observers still contain only positive p. The proof goes as follows. The Bogoliubov coefficients are still determined by taking the inverse Fourier transform of $\tilde{\varphi}_\lambda^{in}$, given by eq. (5.29) wherein one puts $A = 1/\sqrt{4\pi\lambda}$, and $B = 0$. Then, one sees that for $|w| \gg 1/\omega_c$, the integral is dominated by values of p in the cis-Planckian domain $p \ll \omega_c$. Irrespectively of the nature of corrections to the dispersion relation encoded in $g(p)$, this locality implies that for these 'large' $|w|$, one recovers the un-modified propagation on each side of the horizon which characterizes the out modes. This in turn implies that one also recovers the usual Bogoluibov coefficients, for more details see [16].

In order to discuss the third point one must choose the deviation from linearity in $g(p)$. For our purpose, it is sufficient to consider $g(p)$ of the form $g(p) = p(1+\xi p^2/\omega_c^2)$. For $\xi = 1$ one obtains sub-luminous propagation, for $\xi = -1$ super-luminous propagation and for $\xi = i$ one gets dissipation without significant dispersion. These results are easily reached by constructing wave packets and analysing the locus where their phase is stationary. The main point is the following: when the phase of the function varies faster (slower) than the unmodified phase given by $\lambda \ln w$, one has sub-luminous (supra-luminous) propagation. In anticipation to what we shall get for metric fluctuations, we say a few more words in the case of dissipation. In this case, the first order deviation for $\lambda \gg 1$ is of the form

$$
\begin{aligned}
\tilde{\varphi}_\lambda^{in}(p) &= \frac{1}{\sqrt{4\pi\lambda}}\left\{ \frac{\exp\left(-i\lambda \int^p dp' \frac{1}{p'(1+ip'^2/\omega_c^2)}\right)}{p(1 + ip^2/\omega_c^2)} \right\} \\
&\simeq \frac{p^{-i\lambda-1}}{\sqrt{4\pi\lambda}}e^{-\lambda p^2/2\omega_c^2}\left(1 + O(p^2/\omega_c^2)\right)
\end{aligned}
\tag{5.30}
$$

This essentially corresponds to that will be obtained in the case of metric fluctuations. The fact that dissipation (and not only dispersion) should physically occur for black hole was discussed in [16] and the space time image of a packet built with waves of the type (5.30) was schematically presented in Fig. 5, see also [34] for a super-luminous dispersion relation which leads to an effective dissipation of the wave packet.

There is one more point which should be discussed. It concerns the physical relevance of considering propagation *backward* in time. A few remarks may illustrate its relevance[6]. First backward propagation provides the simplest tool to investigate the nature of the field configurations which give rise to Hawking radiation. In particular, it clearly establishes that Hawking quanta emerge from trans-Planckian configurations when one uses,

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[6]However in the absence of a manageable theory of quantum gravity, we are still missing an explicit computation of backreaction effects which will settle this question in unambiguous terms.

as Hawking originaly did, the free Dalembertian for propagating the modes. Secondly, when considering S-matrix elements in a quantum dynamical framework, backward propagation always occurs since one also fixes the final state of the field, see [13, 14]. The simplest example is provided the Feynman *in-out* Green function. This function can be easily related to *in* modes characterized by an *out*-energy λ. Indeed, one has

$$
\begin{aligned}
\alpha_\lambda^{-1}\left(\varphi_\lambda^{in}(w)\right)^* & = \int du\, e^{i\lambda u}\, \frac{<0, out|\varphi(u, v = \infty)\varphi(w, v = 0)|0, in>}{<0, out|0, in>} \\
& = \int du\, e^{i\lambda u}\, G_{\text{in-out}}(u, v = \infty; w, v = 0) \quad (5.31)
\end{aligned}
$$

Because of its out-frequency content, the wave function is global in the sense that it entails the propagation from $\mathcal{J}^+$ to the point where it is evaluated, here $(w = 2r - 1, v = 0)$. In particular when computed in a fluctuating geometry, this function encodes the effects of the metric fluctuations one encounters from $\mathcal{J}^+$ to the point where it is evaluated.

5.2   Backward propagation of *out*-modes

We now have all the tools to perform the comparison. To be as close as possible to what we just presented, we consider the image on $v = 0$ of the monochromatic wave of out-frequency λ. That is, the image on $\mathcal{J}^+$ is $\boldsymbol{\Phi}_{(\lambda)}^+(y) = \varphi_\lambda^{out}(y = w)$ given in eq. (5.24).

Then we have to face the problem mentioned after eq. (3.44): $\varphi_\lambda^{out}(y)$ is not a smooth function but a distribution. This is of course a manisfestation of the trans-Planckian problem. For instance, its derivative with respect to y is ill defined on $y = 0$. Therefore, one must regularize φ_λ^{out} before applying the scattering operator D on it.

The simplest way to define the action of $\ll \mathsf{D} \gg$ on our *out*-function is to work in the momentum conjugated to w. Indeed, the Fourier transform of φ_λ^{out} is well defined in the high p regime, see eq. (5.27). Then using eq. (3.43) we simpy get

$$
\ll \Phi_{(\lambda)}^-(p) \gg = \exp\left(-\frac{p^2\sigma_{\text{eff}}^2}{2}\right) \boldsymbol{\Phi}_{(\lambda)}^+(p) = \exp\left(-\frac{p^2\sigma_{\text{eff}}^2}{2}\right) \frac{\Gamma(i\lambda + 1)}{\sqrt{8\pi^2\lambda}} (\epsilon + ip)^{-i\lambda - 1}
$$

$$
= \exp\left(-\frac{p^2\sigma_{\text{eff}}^2}{2}\right) \frac{(-i)\Gamma(i\lambda + 1)}{\sqrt{8\pi^2\lambda}} \left[e^{\pi\lambda/2}\vartheta(p)p^{-i\lambda - 1} - e^{-\pi\lambda/2}\vartheta(-p)(-p)^{-i\lambda - 1}\right]. \quad (5.32)
$$

Since the effect of the stochastic fluctuations is to multiply the wave function by an even function in p, the relative weight encoding the Bogoliubov coefficients is unaffected. This guarantees that the vacuum state with respect to $p > 0$ leads to the usual properties of Hawking radiation, thereby proving point 2 above. Moreover, eq. (5.32) confirms that the trans-Planckian problem is tamed: The high frequency content, i.e. the near horizon behaviour, is suppressed by a Gaussian factor, as for the *dissipative* case in the former Section, see eq. (5.30). Finally, for $p^2\sigma_{\text{eff}} \ll 1$, one recovers the usual expression of out-modes in terms of p given in (5.27), thereby proving point 1 above.

To complete our analysis, we shall now determine the behaviour of $\ll \Phi_{(\lambda)}^- \gg$ in spacetime. To this end, we inverse Fourier transform separately the two terms (positive and negative p) which appear in eq. (5.32). Using eq. (1.11) in [30], we obtain

$$
\ll \Phi_{(\lambda)}^-(w) \gg = \frac{\sigma_{\text{eff}}^{i\lambda}}{\sqrt{4\pi\lambda}} Z_\lambda(w/\sigma_{\text{eff}}), \quad (5.33)
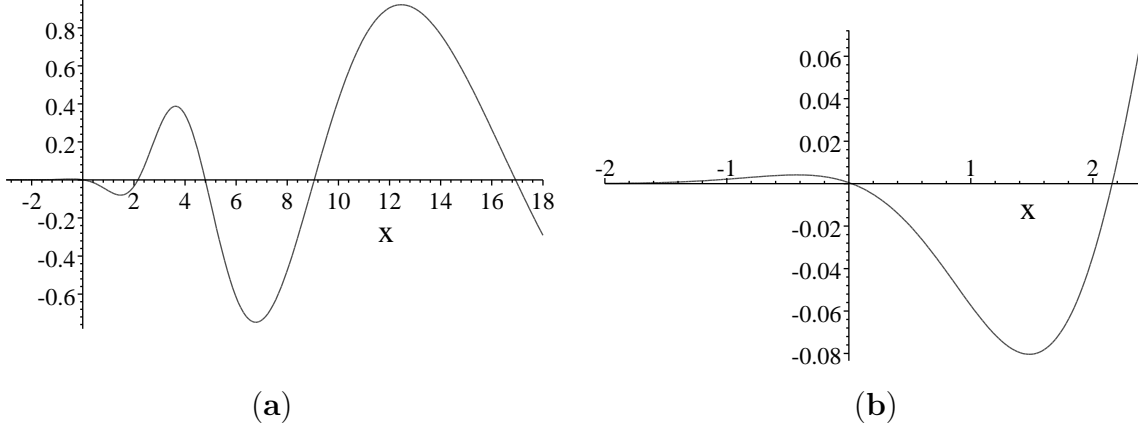$$

(a) **(b)**

Figure 3: Plot (a) gives the real part of $Z_\lambda(x)$ as a function of x for $\lambda = 5.0$. Plot (b) gives an higher resolution of the same function in the interval $-2.0 < x < 2.0$.

where

$$Z_\lambda(x) = \frac{e^{-x^2/4}}{2\sinh(\pi\lambda)} \left[e^{\pi\lambda/2} D_{i\lambda}(\epsilon - ix) - e^{-\pi\lambda/2} D_{i\lambda}(\epsilon + ix) \right] . \tag{5.34}$$

Here $D_\mu(z)$ is the parabolic cylinder function. This function is related to the Whittaker function $W_{\kappa,\nu}(x)$ (see, e.g. [38], p.39)

$$D_\mu(z) = 2^{\mu/2} \left(\frac{z^2}{2} \right)^{-1/4} W_{\mu/2+1/4,1/4} \left(\frac{z^2}{2} \right) . \tag{5.35}$$

In (5.34) we have introduced ϵ positive and infinitesimal in order to specify the phase of $\epsilon - ix$ for positive and negative x. Figure 3 illustrates the behavior of $Z_\lambda(x)$. We have plotted the real part of $Z_\lambda(x)$ using Maple and relation (5.35).

The function $D_{i\lambda}(z)$ has the following asymptotic behavior for $|z| \gg |\lambda|$, see [37],

$$D_{i\lambda}(z) \sim z^{i\lambda} e^{-z^2/4} \left\{ 1 + \frac{\lambda(\lambda + i)}{2z^2} + O(z^{-4}) \right\} , \quad \text{for } -\frac{\pi}{2} < \arg z < \frac{\pi}{2} \quad . \tag{5.36}$$

Using this relation, one verifies that for large negative values of $x = w/\sigma_{\text{eff}}$, Z_λ vanishes. Instead, for large positive z behaves as

$$Z_\lambda(x) \sim x^{i\lambda} \left\{ 1 - \frac{\lambda(\lambda + i)}{2x^2} + O(x^{-4}) \right\} , \tag{5.37}$$

The limit $x = w/\sigma_{\text{eff}} \to \infty$ corresponds to the far from horizon region or to $\sigma_{\text{eff}} \to 0$. The last case corresponds to no metric fluctuations. In this regime (5.37) reproduces the unperturbed out wave function given in eq. (5.24). The plot of the unperturbed out-wave function $x^{i\lambda}$ is shown in Figure 4. By comparing Figures 3 and 4 one clearly sees that metric fluctuations strongly affect the back scattered wave for values of w close to smeared horizon. Indeed, close to the horizon, i.e. for $x^2 \ll \lambda$, the function Z_λ behaves as, see [37],

$$Z_\lambda(x) = \frac{2^{i\lambda/2}}{2\sqrt{\pi}} \Gamma\left(\frac{1 + i\lambda}{2} \right) \left\{ 1 + \coth(\pi\lambda/2)e^{i\pi/4} x\sqrt{\lambda - i/2} + O(x^2\lambda) \right\} \tag{5.38}$$
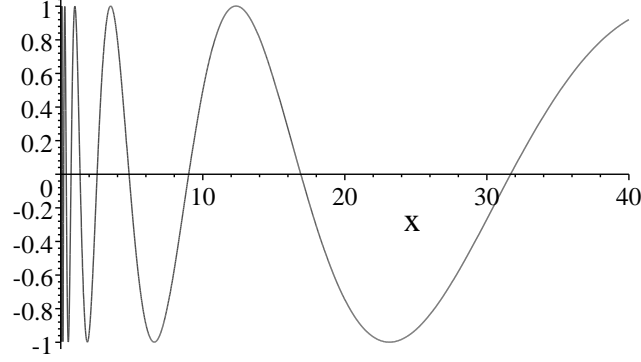
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Figure 4: Plot of the real part of the unperturbed out-wave $x^{i\lambda}$ for $\lambda = 5$.

The most important feature is the disappearance of the infinite trans-Planckian reservoir of fluctuations of equal amplitude which characterizes the unperturbed out-wave $w^{i\lambda}$. This implies that backward propagation stops for any localized wave packet. In order words, the amplitude of the wave packet is dissipated in its backward motion once the wave packet enters the near horizon region $x < \lambda$.

To show this, let us consider a rapidly modulated Gaussian wave packet which on $\mathcal{J}^+$ has the image

$$\Phi_{\bar\lambda}^+(u) = e^{-i\bar\lambda(u-u_0)} e^{\frac{-(u-u_0)^2}{2b^2}} \, . \tag{5.39}$$

This image is localized around $u = u_0$ and has width b. In the limit $b \to \infty$ one obtains a monochromatic out-plane wave of frequency $\bar\lambda$. Its Fourier decomposition is given by

$$\Phi_{\bar\lambda}^+(u) = \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda \, e^{-i(u-u_0)\lambda} \, e^{-b^2(\lambda-\bar\lambda)^2/2} \, . \tag{5.40}$$

We first consider its backward propagation in the usual non-fluctuating geometry. In this case, the plane wave $e^{-i\lambda u}$ takes the form $w^{i\lambda}$ on $v = 0$. Thus the image on $v = 0$ of the wave-packet (5.39) is

$$\Phi_{\bar\lambda}^-(w) = e^{i\bar\lambda(\ln w + u_0)} e^{-(\ln w + u_0)^2/(2b^2)} \, . \tag{5.41}$$

For late time u_0, it hugs the horizon $w = 0$ since it is centered around $w = e^{-u_0}$. Moreover it is highly blue-shifted since its mean frequency is equal to $\bar\lambda e^{u_0}$ for an infalling observer. Finally, no matter how large u_0 is, its maximum amplitude is still 1. There is no dissipation. This is the trans-Planckian problem.

Let us now analyze how the fluctuating geometry changes this picture. In the presence of fluctuations, the image on $v = 0$ of the plane wave $e^{-i\lambda u}$ is $\sigma_{\text{eff}}^{i\lambda} Z_\lambda(w/\sigma_{\text{eff}})$ instead of $w^{i\lambda}$. For simplicity, we first consider a wave packet with $\bar\lambda \gg 1$ and $b \gg 1$. The first condition means that it is rapidly oscillating (in the units of the surface gravity) whereas the second means that its frequency content is well peaked around the mean frequency $\bar\lambda$. Using the near horizon behavior of Z_λ, eq. (5.38), the scattered wave packet is

$$\begin{aligned} \ll \Phi_{\bar\lambda}^-(w) \gg &= \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda \, e^{iu_0\lambda} \, e^{-b^2(\lambda-\bar\lambda)^2/2} \sigma_{\text{eff}}^{i\lambda} Z_\lambda(w/\sigma_{\text{eff}}) \tag{5.42} \\ &\simeq \frac{b}{\sqrt{8\pi}} \int_{-\infty}^{\infty} d\lambda \, e^{i\lambda(u_0 + \ln\sigma_{\text{eff}} + \ln(\lambda)/2)} \, e^{-b^2(\lambda-\bar\lambda)^2/2} \left\{ 1 + \frac{w}{\sigma_{\text{eff}}} e^{i\pi/4} \sqrt{\lambda} \right\} \, . \end{aligned}$$

In the second line we have dropped all irrelevant phases and we have used the asymptotic behavior of the Γ function and the coth, in anticipation to the fact that the main contribution of the integral will arise from values of λ centered around $\bar{\lambda}$.

By performing the integral by a saddle point approximation (this is perfectly valid when $b^2 \gg 1$), one obtains the following behavior

$$\ll \Phi_{\bar{\lambda}}^-(w) \gg \simeq \exp\left(-\frac{(u_0 + \ln \sigma_{\text{eff}} + \frac{1}{2}\ln \bar{\lambda})^2}{2b^2}\right)\left\{1 + \frac{w}{\sigma_{\text{eff}}}e^{i\pi/4}\sqrt{\bar{\lambda}}\right\}. \qquad (5.43)$$

Hence once $u_0 > -\ln \sigma_{\text{eff}}$, the image on $v = 0$ vanishes like a Gaussian in u_0.

This strong dissipation is only valid for tight wave packets in λ, i.e. for $b \gg 1$. Instead, in the opposite regime $b \ll 1$, for tight wave packets in position space, the dissipation is milder. Indeed, in the limit $b \to 0$, the Gaussian factor in eq. (5.42) can be ignored in the large u_0 limit. Then, the main contribution comes from the first pole of the Γ function in the positive imaginary λ axis. In this regime the decrease of the wave packet is given by $e^{-(u_0 + \ln \sigma_{\text{eff}})}$.

In brief, as long as the mean position in w on $v = 0$ of the wave packet is larger than $\lambda\sigma_{\text{eff}}$, its image is unaffected by the metric fluctuations since Z_λ still behaves as $w^{i\lambda}$. Instead, once it enters the near horizon region, its amplitude rapidly decreases.

6   Conclusion

It is perhaps appropriate to list what we learn from our analysis.

1. When the relative width of the smeared horizon is small enough, i.e. when $\delta r_{EH}/r_{EH} \simeq \sigma_{\text{eff}} \ll 1$, metric fluctuations in the near horizon geometry affect the asymptotic properties of Hawking radiation only slightly, in the second order of σ_{eff}, see (4.17).

2. The reason for this stability can be seen from the short distance behaviour of the *in*-Green function, see (4.7). Indeed, for $|\Delta y| \ll 1$, one recuperates the usual Hadamard behavior which guarantees point 1.

3. Backward propagated wave packets representing Hawking quanta of energy λ are dissipated when their Doppler shift frequency $i\partial_r$ reaches σ_{eff}^{-1}, i.e. when their separation in r from the event horizon approaches $\lambda\sigma_{\text{eff}}$, see (5.42).

The attentive reader will notice that points 2 and 3 are rather difficult to conciliate. Indeed if the behavior of the in-Green function is not modified why for is point 3 relevant? In other terms what is the high energy behavior of the theory? Unmodified as suggested by point 2 or dramatically modified as indicated by point 3?

To answer these questions one should first reconsider the domain of validity of the scheme we used. Our scheme is based on eq. (3.2) which represents the free propagation of a test-field in a fluctuating geometry. This is an approximative description which completely neglects the back-reaction effects induced by the field φ itself. In other words we are working in the regime when the metric fluctuations are not significantly affected by the energy density carried by φ. Therefore, our main hypothesis is that there is an *intermediate* regime in which the metric fluctuations induced by all the other degrees of freedom cannot be neglected whereas the back-reaction effects due to φ can be neglected.

For frequencies below this regime, we have proven that the metric fluctuations play no role, see eq. (3.41). For higher frequencies, we cannot say much. Nevertheless it is most probable that the stochastic description we used fails. Therefore we can trust the behavior of the in-Green function only for separations within the intermediate domain.

Having clarified this point, we can now explain why the high frequency behavior of the in-Green function differs so much from that of backward scattered waves. The reason is the following. Being a function of the difference in V_ϕ, the in-Green function is hardly sensitive to the metric fluctuations in the coincidence point limit, see eq. (4.7). On the contrary, as emphasized after eq. (5.31), backscattered wave functions defined on $\mathcal{J}^+$ are sensitive to the metric fluctuations they have encountered when evaluated near the horizon. Moreover, since their frequency is blue shifted, they are inevitably strongly affected by the metric fluctuations.

If this explains that the behaviour of the in-Green is perfectly compatible with that of backscattered waves, it does not tell us what happens to these waves and their energy density when their amplitudes diminish. To describe the fate of their energy density one must consider the dynamics of the degrees of freedom which engender the metric fluctuations. Indeed, one must go beyond the stochastic treatment of these fluctuations in order to be able to describe the trans-Planckian momentum recoils which shall *inevitably* be induced by Hawking photons when they are traced backward near the event horizon, see [14, 35] for preliminary attemps to describe this physics.

In brief, the main outcome of the paper is to have provided physical foundations in terms of metric fluctuations to the concept of effective propagation of light near a black hole horizon.

This allows to address in a rational scheme the question of the domain of validity of this effective propagation. It also provides an explanation for the vexing question of the apparent violation of local Lorentz invariance[9, 10, 16]. The neatest way to characterize this violation is to focus on the near horizon behavior of a monochromatic mode ϕ_λ^{out}. In the absence of modification of the dispersion relation, this mode behaves as $w^{i\lambda}$ where $w = 2r - 1$. Hence there is no length which allows one to distinguish low from high momenta. This absence is a consequence of the local Lorentz invariance of theories based on the usual Dalembertian. On the contrary, when dealing with a modified dispersion relation, one breaks this invariance since the new dynamical equation is written in a preferred frame. For acoustic black holes this makes good sense since both the frame and the critical lenght, which characterizes what "high" frequency means, are given by the constituents of the fluid. On the contrary, it is rather unclear to see the origin of such a preferred frame for a gravitational black hole. One of the main virtues of the present work is to provides a simple answer to this puzzle. Indeed the ensemble of metric fluctuations unambiguously determines, σ, the constant spread in r (measured along $v = 0$) of the distribution of the backward propagated rays representing the event horizon. Because of the hypothesis of stationarity metric fluctuations, the modified equation governing light propagation has a simple and stationary expression in the v, r coordinate system. In particular, the cut-off lenght σ appears only through powers of $\sigma \partial_r |_v$. In this case what might be interpreted as the origin of a "violation of Lorentz invariance" originates from the ensemble of stationary metric fluctuations.

A   The leading non-linear corrections

In this Appendix we shall demonstrate the validity of using the linearized and simplified expression, eq. (2.20), of $V_\phi(u)$ for obtaining the dominant non-linear effects in w_0 on the propagation of the field described by eq. (3.9).

To simplify the analysis, we consider the Fourier transform with respect to w of the image of the pulse defined on $\mathcal{J}^-$ evaluated in a given realization:

$$\Delta_\phi^+(u|p) = \frac{1}{\sqrt{2\pi}} \int dw\, e^{ipw} \Delta_\phi^+(u|w)\,. \tag{A.1}$$

Using eq. (3.7), the mean image is given by

$$\bar{\Delta}^+(u|p) = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\phi\, e^{ipV_\phi(u)}\,, \tag{A.2}$$

the integral over w being trivial.

The analysis of the null ray propagation in a fluctuating geometry shows [7] that the complete expression for $V_\phi(u)$ is of the form

$$V_\phi(u) = -1 - e^{-u} + F_0 + F_1 e^{-u} + O(e^{-2u})\,, \tag{A.3}$$

where the correction terms F_0 and F_1 are series in w_0. The corrections terms multiplied by e^{-u} or higher powers of it are not important in the late time regime since they become arbitrarily smaller than those of F_0. Hence we can drop F_1. The remaining correction term F_0 fixes the precise value of the image on $\mathcal{J}^-$ of the exact horizon in every realization of the fluctuating geometry. It also determines the late times corrections to $\bar{\Delta}^+(u|p)$. It has the following expansion in powers of w_0

$$F_0 = w_0 \sin(\phi + \phi_0) + w_0^2 f_1(\omega) \sin(2\phi + \phi_1) + O(w_0^3)\,. \tag{A.4}$$

Let us substitute (A.4) into (A.2) and show that the leading non-linear corrections to $\bar{\Delta}^+$ all come from the linear term in w_0.

In the low p regime, i.e. for $p \ll 1/w_0$, all correction terms are negligeable since $w_0 \ll 1$ in our dimensionless units. Upon reaching the high p regime, for $p \simeq 1/w_0$, all terms of the form pw_0^n for $n \geq 2$ are still very small. Using this fact we can rewrite (A.2) as

$$\bar{\Delta}^+(u|p) = \frac{e^{-ip(1+e^{-u})}}{(2\pi)^{3/2}} \times \int_0^{2\pi} d\phi\, e^{ipw_0 \sin\phi} \left[1 + ipw_0^2 f_1(\omega) \sin(2\phi + \phi_1) + \ldots\right]$$

$$= \frac{e^{-ip(1+e^{-u})}}{(2\pi)^{3/2}} \times \left[G_0(pw_0) + pw_0^2 G_1(pw_0) + pw_0^3 G_2(pw_0) + \ldots\right]\,. \tag{A.5}$$

The form of the integrals implies that functions $G_n(pw_0)$ are regular. For $pw_0 = 0$ they are all finite, while for $pw_0 \to \infty$, $G_n(pw_0)$ are of the same order of magnitude. Therefore the corrections to the leading term G_0 remain small as long as $pw_0^2 \ll 1$. If w_0 is of the order of the Planck length, the regime of validity of the approximation, $p \ll w_0^{-2}$, goes far beyond the Planckian scale. Having established this result, we can now make more precise the neglection of the F_1 term in eq. (A.3) in the late time regime. It is sufficient to have $e^{-u} < w_0$, since the linear term in w_0 in F_1 behaves in this regime like the quadratic term of F_0.

In other words, at late times and for all the regimes starting from small p till momenta much higher than the Planck one, $G_0(pw_0)$ $(= 2\pi J_0(pw_0)$, see (3.30)) gives the leading non-linear corrections to $\bar{\Delta}^+(u|p)$. This proves the claim since $G_0(pw_0)$ entirely comes from the linear term in F_0.

B   Backward Scattering

In this Appendix we discuss the properties of the backscattered wave of out-frequency λ when the average is performed over the phase only.

Its image on $\mathcal{J}^-$ or on $v = 0$, $\bar{\Phi}_{(\lambda)}^-$, can be calculated by making a Fourier transform as it was done for $\ll \Phi_{(\lambda)}^-(p) \gg$ in Section 5. Here, we shall offer an alternative way to compute this image. We start by regularizing the out wave $\varphi_\lambda^{out}(y)$, see eq. (5.24). One can view this distribution as the limit $\epsilon \to 0$ on the real $w = y$ axis of the difference of two analytic functions. Explicitly one has

$$\sqrt{4\pi\lambda}\,\varphi_\lambda^{out}(w) = \vartheta(w)w^{i\lambda} = \frac{1}{2\sinh(\pi\lambda)}\left(e^{\pi\lambda/2}(\epsilon - iw)^{i\lambda} - e^{-\pi\lambda/2}(\epsilon + iw)^{i\lambda}\right) \qquad \text{(B.1)}$$

Notice that the out-mode has been expressed by a Bogoliubov relation as the difference of two in-modes which are perfectly well defined on the horizon. We can then safely apply $\bar{D}$ to each of them. We shall make use of the relation

$$P_\nu^{-\mu}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + 1)(\rho^2 + z^2)^{\nu/2}}J_\mu\left(\rho\frac{\partial}{\partial z}\right)z^\nu, \qquad \text{(B.2)}$$

where $P_\nu^{-\mu}$ denotes the generalized Legendre function. Eq. (B.2) was derived by Filon in 1903 [25] and cited in the book by Watson [26], p.51, who proposed to the reader to prove this formula as an exercise! Using this equation, we obtain

$$\bar{\Phi}_{(\lambda)}^-(w) = J_0(iw_0\partial_w)\varphi_\lambda^{out}(w)$$
$$= \frac{1}{\sqrt{4\pi\lambda}}\frac{1}{2\sinh(\pi\lambda)}\left\{e^{\pi\lambda/2}\left[(\epsilon - iw)^2 + w_0^2\right]^{i\lambda/2}P_{i\lambda}\left(\frac{\epsilon - iw}{\sqrt{(\epsilon - iw)^2 + w_0^2}}\right)\right.$$
$$\left. -e^{-\pi\lambda/2}\{-iw \to +iw\}\right\}, \qquad \text{(B.3)}$$

where $P_\mu(z)$ is the Legendre function. We have introduced ϵ positive and infinitesimal in order to specify the phase of $\sqrt{(\epsilon - iw)^2 + w_0^2}$ in the three sectors: $|w| < w_0$ and $|w| > w_0$.

Before presenting the detailed behavior of $\bar{\Phi}_{(\lambda)}^-$ a few simple deductions can be made. First, in the limit $w_0 \to 0$, $\bar{\Phi}_{(\lambda)}^-$ reduces to the usual out wave function given in eq. (5.24)

since $P_{i\lambda}(1) = 1$. Secondly, one finds, as one should do, that $\bar{\Phi}^-_{(\lambda)}(w)$ vanishes for $w < w_0$ since $\Phi^+_{(\lambda)}(y)$ vanished for negative y. Thirdly, the spread of the effective horizon $w = \pm w_0$ clearly appears in $\bar{\Phi}^-_{(\lambda)}$.

Figure 5 illustrates the behavior of $\bar{\Phi}^-_{(\lambda)}$ for $\lambda = 5$. This behavior is typical for values λ greater than 1. For smaller frequencies, the amplitude of the oscillations between $\pm w_0$ is much smaller than 1. We have presented only the real part of $\bar{\Phi}^-_{(\lambda)}$ since the imaginary part behaves similarly.

The main features of $\bar{\Phi}^-_{(\lambda)}$ directly follow from its properties near the singular points $w = \infty$ and $w = \pm w_0$. The first manifestation of the metric fluctuations shows up around $w = 20w_0$ where the amplitude of the oscillations start to decrease, see (fig. 5a). This modulation of the amplitude arises from the correction term of the asymptotic behavior of $\bar{\Phi}^-_{(\lambda)}$. Using eq. 8.1.2 in [37], one has

$$\bar{\Phi}^-_{(\lambda)}(w) = \frac{1}{\sqrt{4\pi\lambda}} w^{-i\lambda} \left\{ 1 - \frac{\lambda(\lambda+i)}{4} \left(\frac{w_0}{w}\right)^2 + O\left(\frac{w_0}{w}\right)^4 \right\} \tag{B.4}$$

The second effect is located for values of w slightly bigger than w_0, see (fig. 5b). One has a rapidly oscillating function with a decreasing amplitude. The origin of this structure can be understood from eq. (3.9): the logarithmicly divergent behavior of φ_λ^{out} at $w = 0$ has been shifted with some coherence to $w = w_0$ because of the stationarity of the shift with respect to ϕ at its maximal value. This behavior can be seen from the analytical property of the function. Using eq. 8.1.5 in [37], one finds that $\bar{\Phi}^-_{(\lambda)}$ is a sum of two hypergeometric functions. The first one controls the overall shape of the function near $w = w_0$ whereas the second one provides the rapidly oscillating behaviour. In the limit $w \to w_0$ one has

$$\bar{\Phi}^-_{(\lambda)}(w) = \frac{1}{\sqrt{4\pi\lambda}} w^{-i\lambda} \left\{ A_1 + A_2 \left(1 - \frac{w^2}{w_0^2}\right)^{1/2+i\lambda} + O\left(1 - \frac{w^2}{w_0^2}\right) \right\} \tag{B.5}$$
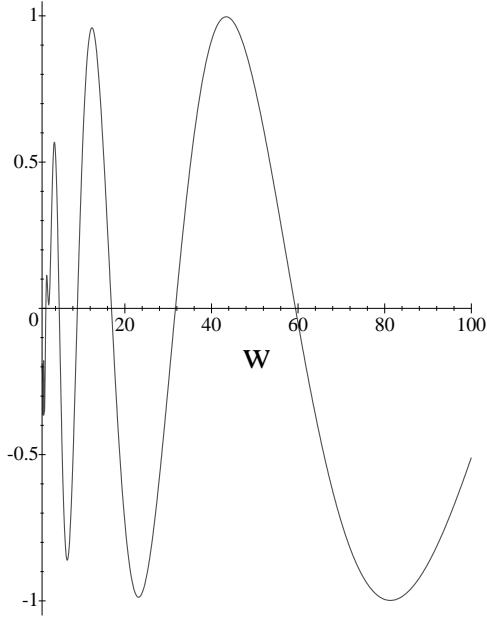
where $A_1 = 2^{i\lambda}\pi^{-1/2}\Gamma(i\lambda + 1/2)\Gamma^{-1}(i\lambda + 1)$ and $A_2 = 2^{-i\lambda-1}\Gamma(-i\lambda - 1/2)\Gamma^{-1}(i\lambda)$. Notice that one recovers arbitrarily high frequencies in this behavior. Therefore one can fear that one has simply shifted the trans-Planckian problem from $w = 0$ to $w = w_0$. This is not the case for two reasons. First the amplitude of these oscillations decreases as $\sqrt{w^2 - w_0^2}$ for $w \to w_0$. Therefore the amplitude to find high frequencies decreases. Secondly, when one averages over the amplitude w_0, this structure is erased, as shown in Figure 4. Therefore, it is an artifact of our simplified averaging procedure.

The third salient fact is that on the other side of w_0 no rapid oscillations are found, see (figs. 5b-c). There are nevertheless oscillations but, as indicated in the next equation (obtained again from eq. 8.1.5 in [37]) they are suppressed by $\sinh(\pi\lambda)$ with $\lambda = 5$.
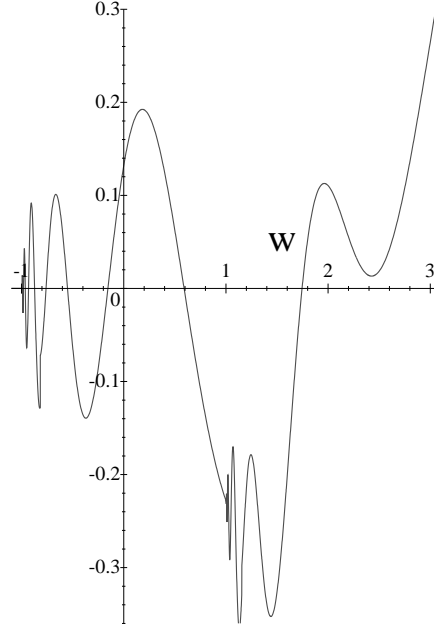
$$\bar{\Phi}^-_{(\lambda)}(w) = \frac{1}{\sqrt{4\pi\lambda}} w^{-i\lambda} \left\{ A_1 + i\frac{A_2}{\sinh(\pi\lambda)} \left(\frac{w^2}{w_0^2} - 1\right)^{1/2+i\lambda} + O\left(\frac{w^2}{w_0^2} - 1\right) \right\} \tag{B.6}$$

The fourth point is that $\bar{\Phi}^-_{(\lambda)}$ is regular around $w = 0$, see (fig. 5d). Analytically one has
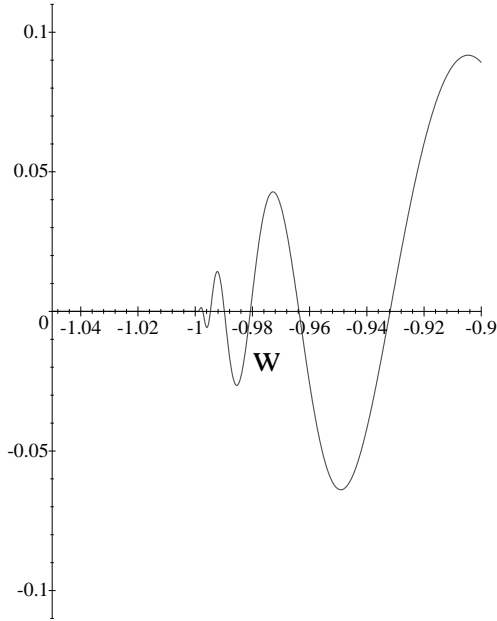
$$\bar{\Phi}^-_{(\lambda)}(w) = \frac{1}{\sqrt{4\pi\lambda}} w_0^{-i\lambda} \left\{ \tilde{A}_1 + \tilde{A}_2 \frac{w}{w_0} + O\left(\frac{w^2}{w_0^2}\right) \right\} \tag{B.7}$$
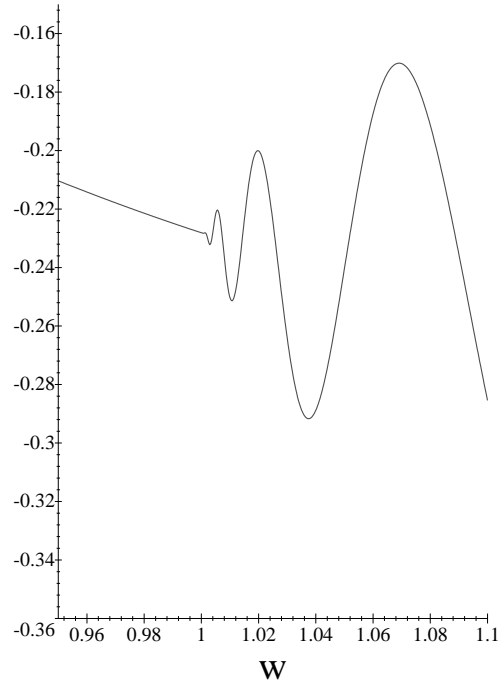
(**a**)

(**b**)

(**c**)

(**d**)

Figure 5: These figures illustrate the salient properties of $\bar{\Phi}^-_{(\lambda)}(w)$. We have plotted the real part of $\sqrt{4\pi\lambda}\,\bar{\Phi}^-_{(\lambda)}$ as a function of w/w_0 for $\lambda = 5.0$ for 4 different sectors of w.

where $\tilde{A}_1 = \pi^{-1/2}\Gamma(\frac{1}{2}+\frac{i\lambda}{2})\Gamma^{-1}(1+\frac{i\lambda}{2})$ and $\tilde{A}_2 = -\frac{i\lambda}{2}\pi^{-1/2}\Gamma(\frac{i\lambda}{2})\Gamma^{-1}(\frac{1}{2}+\frac{i\lambda}{2})$. Therefore, the trans-Planckian reservoir of oscillations which was present in φ_λ^{out} has been eliminated. This is the main physical result.

The fifth point concerns the structure when w approaches $-w_0$ from above. Analytically, one has

$$\bar{\Phi}_{(\lambda)}^{-}(w) = \frac{1}{\sqrt{4\pi\lambda}}(-w)^{-i\lambda}\left\{-i\frac{A_2}{\tanh(\pi\lambda)}\left(\frac{w^2}{w_0^2}-1\right)^{1/2+i\lambda} + O\left(\frac{w^2}{w_0^2}-1\right)\right\} \qquad \text{(B.8)}$$

As in eq. (B.5), the amplitude of the fluctuations deceases like a square root when approaching the boundary $w = -w_0$. As already mentioned, for smaller values of w one finds identically zero. One can apply to this fifth point the remarks made in the second point.

In brief, the main physical result is the fourth point. Moreover, in contradistinction with the second, third and fifth points, it is stable if one considers an ensemble of amplitudes w_0. The physical consequence of the disparition of the reservoir of oscillations for $w < w_0$ is that the backward propagation of any wave packet will stop around w_0. See Section 5.2 for more details.

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